Continuum Model for River Networks

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The effects of erosion, avalanching, and random precipitation are captured in a simple stochastic partial differential equation for modeling the evolution of river networks. Our model leads to a self-organized structured landscape and to abstraction and piracy of the smaller tributaries as the evolution proceeds. An algebraic distribution of the average basin areas and a power law relationship between the drainage basin area and the river length are found.

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A fractal river network is a striking example of self-organized criticality. The physics of river network evolution arises from an interplay of the structured landscape governing the water flow with the erosional effects of the water feeding back into further sculpting of the landscape. Extensive studies of the fractal characteristics of real river networks have been carried out [1–7]. Hack [2] has studied the relationship between the length of a river l and the area of a drainage basin s. s is a measure of the total area of the land covered by the principal stream and its tributaries that feed into the network. Hack's measurements indicate that for basin areas s ranging over almost five decades (up to 375 square miles), $s \sim l^{\phi}$ with the exponent $1/\phi \sim 0.57$. Other measurements of the distribution of drainage basin areas suggest a power law scaling of the form $P(s) \sim s^{-\tau}$ with $\tau = 1.45 \pm 0.03$ [1].

Most of the models of river networks fall into two categories. The first is restricted to reproducing the statistical properties of networks [8]. More recently, models for the *evolution* of river networks have been developed. Based on careful studies of river data [1], Rinaldo *et al.* [9] have suggested that the effects of local optimal rules equivalent to critical erosion parameters lead to statistical characteristics for the networks similar to global constraints of minimum energy dissipation. Leheny and Nagel [10] introduced a lattice model that incorporated erosion and showed a competition in growth between neighboring river basins relevant for late stages of evolution. Kramer and Marder [11] have also constructed a lattice model that allows for the elucidation of the scaling properties of the large scale features of river networks. In addition, they have proposed coupled differential equations for two scalar fields, the height of the soil and the depth of the water flowing over the soil. An analysis of these equations has led to an understanding to the shape and stability of individual river channels.

Our principal goal is to introduce and numerically study a simple stochastic partial differential equation for the evolution of the landscape of a river network. Our model is a field theory for the soil height $h(\mathbf{x}, t)$ and takes into account the effects of random precipitation, erosion, and the avalanching of soil. An initially smooth landscape evolves into a nontrivial spatially self-organized state in which Hack's law and the algebraic distribution of drainage basin areas are obtained.

The evolution equation may be written in the compact form

$$(\partial/\partial t)h(\mathbf{x},t) = \mathcal{D} \cdot h(\mathbf{x},t) - k\nabla^4 h(\mathbf{x},t) + \eta(\mathbf{x},t), \tag{1}$$

where $\mathcal{D}\cdot \equiv D_1 \partial_y^2 + \tilde{D}_2(|\nabla h|)\partial_x^2$ and $\mathbf{x} \equiv (x,y)$. The coefficient $\tilde{D}_2(|\nabla h|)$ is defined as

$$\tilde{D}_2[|\nabla h|] = \begin{cases} D_1 > 0 & \text{if } |\nabla h| > M, \\ D_2 < 0 & \text{if } |\nabla h| < M, \end{cases} \tag{2}$$

and the random noise $\eta(\mathbf{x},t) \equiv \varepsilon r(\mathbf{x},t)$ is Gaussian distributed with zero average and correlation

$$\langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle = \varepsilon^2 \delta^d(\mathbf{x},\mathbf{x}')\delta(t-t'). \tag{3}$$

The equation describes the temporal evolution of a two dimensional landscape with periodic boundary conditions in the x direction and a dominant water flow in the y direction due to an incline. The equation allows for ordinary diffusion in the y direction and a stabilizing $\nabla^4 h$ term working as an ultraviolet regulator. In the x direction, a diffusion term is operational as long as $|\nabla h| > M$. These diffusional processes mimic the avalanching effect discussed by Leheny and Nagel: When neighboring sites have large differences in heights, avalanching of the soil occurs, leading to a smoothing effect. The erosional processes are captured by the negative diffusion coefficient D_2 when $|\nabla h| < M$. Such a term accentuates the height difference and captures the physics of erosion—the water flowing in the shallower parts of the landscape erode the soil further, *increasing* the height difference. In the simple version of our model, the erosional processes are blind—the erosion is uncorrelated with the water flow. Also, our analysis has been carried out in the limit of a large underlying y slope so that the water flow is directed downhill and ordinary diffusion is operative in the y direction. Large basins are known to have smaller overall

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slopes so that a less directed water path would be more appropriate [9]. The noise η mimics the erosional effects of random precipitation. It should be noted that this noise is an essential ingredient of the dynamics: Noisy initial conditions would evolve into a flat landscape under only the deterministic part of Eq. (1).

Our equation is a generalization of equations developed in other contexts. In the limit that $D_1 = D_2 > 0$, the equation reduces to the Edwards-Wilkinson equation for linear growth processes [12]. When D_1 and D_2 are both positive but different, the equation is akin to Barenblatt's equation [13] describing the pressure in an elastoplastic porous medium that allows for the contraction and expansion of the medium due to the flow. The novel ingredient in our equation is the possibility of a negative diffusion coefficient. As in sandpile models of self-organized criticality [14], smoothing diffusional processes kick in when the gradient exceeds a critical value. While the sandpile models are driven by the random *addition* of sand, here the instability is caused by the erosion and the noise. Our model evolves in a self-organized manner to a noisy boundary separating stability and instability, as do the sandpile models. In the sandpile case, there is a well-defined interval between the addition of grains during which the relaxation takes place. In contrast, the dynamics is continuously turned on in our model.

In order to discretize Eq. (1) it is convenient to work with dimensionless variables $\tilde{t}, \tilde{\mathbf{x}}, \tilde{h}$ defined as

$$t = \frac{k}{D_1^2}\tilde{t}, \tag{4}$$

$$\mathbf{x} = \sqrt{\frac{k}{D_1}}\tilde{\mathbf{x}}, \tag{5}$$

$$h = \frac{\varepsilon}{\Delta x \sqrt{D_1}}\tilde{h}, \tag{6}$$

where Δx is the lattice spacing of the spatial mesh we will be using. In the following, we shall omit the tildes. The discretized version of the equation is

$$h(\mathbf{x}, t + \Delta t) = h(\mathbf{x}, t) + \frac{\Delta t}{\Delta x^2}\{\Delta_y h(\mathbf{x}, t) + D[|\nabla h|]\Delta_x h(\mathbf{x}, t)\} - \frac{\Delta t}{\Delta x^4}\Delta^2 h(\mathbf{x}, t) + \sqrt{\Delta t}\, r(\mathbf{x}, t). \tag{7}$$

Here Δ_x and Δ_y are discrete second derivatives in the x and y directions and Δ^2 is the discrete square Laplacian defined so that the Fourier transform of $\Delta^2 h$ does not contain anisotropic terms to the leading order, and

$$D[|\nabla h|] = \begin{cases} 1 & \text{if } |\nabla h| > M, \\ D \equiv D_2/D_1 < 0 & \text{if } |\nabla h| < M. \end{cases} \tag{8}$$

Once the landscape reaches a self-organized state, we obtain a measure of s by adding a test drop of precipitation on each site and following its downhill descent along the steepest path [1,10]. A drop at site (x, y) has three possible destination sites—$(x - 1, y + 1)$, $(x, y + 1)$, or $(x + 1, y + 1)$. The site with the smallest h is selected. On a given site, s is defined as the number of drops collected on that site during the downhill evolution.

If we define $P(s, L)$ as the density of sites with a total flow s on a landscape of linear size L, one may combine Hack's law and the algebraic scaling of the river basin areas into a generalized scaling form [15]

$$P(s, L) = s^{-\tau} F(s/L^{\phi'}). \tag{9}$$

Assuming that rivers are fractals with length $l \sim L^{d_F}$, the characteristic basin area $s_c \sim L^{\phi'} \sim l^{\phi'/d_F}$, implying Hack's law with $\phi = \phi'/d_F$. In the present case we find $d_F = 1$, i.e., the rivers are self-affine, and thus $\phi = \phi'$.

From the definition of s, it follows that its average $\sum_s P(s, L)s = (L + 1)/2$, which combined with (9), leads to the scaling law $\phi = (2 - \tau)^{-1}$ [16]. Further, for $\tau > 1$, the normalizability of $P(s, L)$ in the $L \to \infty$ limit imposes a constraint that $\lim_{x \to 0} F(x)$ is universal and equal to $\tau - 1$. Figure 1 shows a plot of $P(s, L)$ for different sizes $L = 10$, 30, 50, and 100 at time $t = 10$. These results were obtained in the regime where $\Delta t/\Delta x^2 \sim 10^{-3}$ well below the linear stability limit $\Delta t/\Delta x^2 \sim 0.25$. The figures are then collapsed into a single scaling plot (Fig. 2) with the choices of $\tau = \frac{4}{3}$ and $\phi = (2 - \tau)^{-1} = \frac{3}{2}$ [17]. Within the error estimates of our analysis, these exponents are the same as obtained in Scheidegger's static model [6] for river networks [18]. They are also within the range observed for real rivers in the small basin limit. It should be mentioned that the above scaling form is expected to be valid only in the large s and large L regime. Long time runs on small

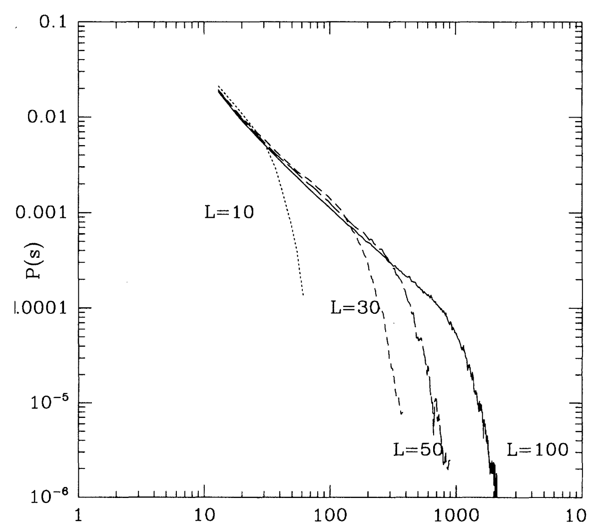

FIG. 1. Plot of $P(s)$ vs s for sizes $L = 10$, 30, 50, and 100 evaluated at the (absolute) time $t = 10$ in $d = 2 + 1$. The values for the parameters are $\Delta t = 0.01, \Delta x = 3.0, M = 1$, and $D_2/D_1 = -0.1$.

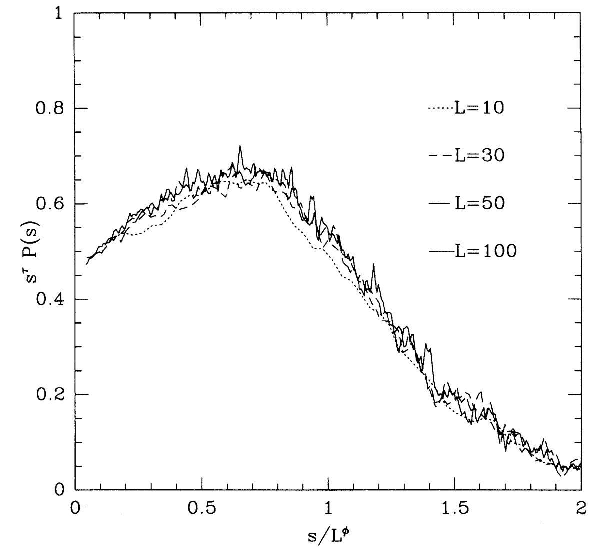

FIG. 2. Collapse of the curves of $P(s)$ from Fig. 1 with $\tau = \frac{4}{3}, \phi = (2 - \tau)^{-1} = \frac{3}{2}$.

systems are suggestive of a non-Scheidegger universality class at large times. However, a quantitative study of this regime appears to be beyond our present capability due to computational limitations.

We have also monitored the temporal evolution of the roughness of the landscape. The roughness W is defined as

$$W(t) = \left((1/L^2) \sum_{\mathbf{x}} \langle [h(\mathbf{x}, t) - \overline{h}(t)]^2 \rangle \right)^{1/2},$$

where $\overline{h}(t) = \sum_{\mathbf{x}} h(\mathbf{x}, t)/L$. The average was performed over 300 samples corresponding to different realizations of the noise. We find an intermediate regime with an algebraic growth $W(t) \sim t^{\beta}$ with an exponent $\beta = 0.21 \pm 0.02$ (Fig. 3). This exponent value is different from the exactly solvable Edwards-Wilkinson model [12] where $\beta = 0$, i.e., $W(t) \sim \sqrt{\ln t}$. At longer times saturation due to finite size is observed. In this context, it is interesting to note the recent work of Czirok, Somfai, and Vicsek [20] on a geomorphological micromodel of mountains. The evolution of river networks is shown in Fig. 4, where

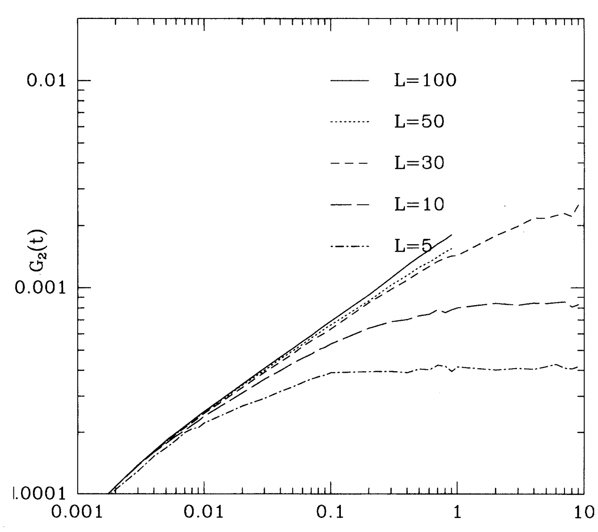

FIG. 3. Temporal evolution of roughness $G_2(t) = W^2(t)$. The parameters used for the simulations were $\Delta t = 10^{-4}, \Delta x = 0.5, M = 1$, and $D_2/D_1 = -0.1$.

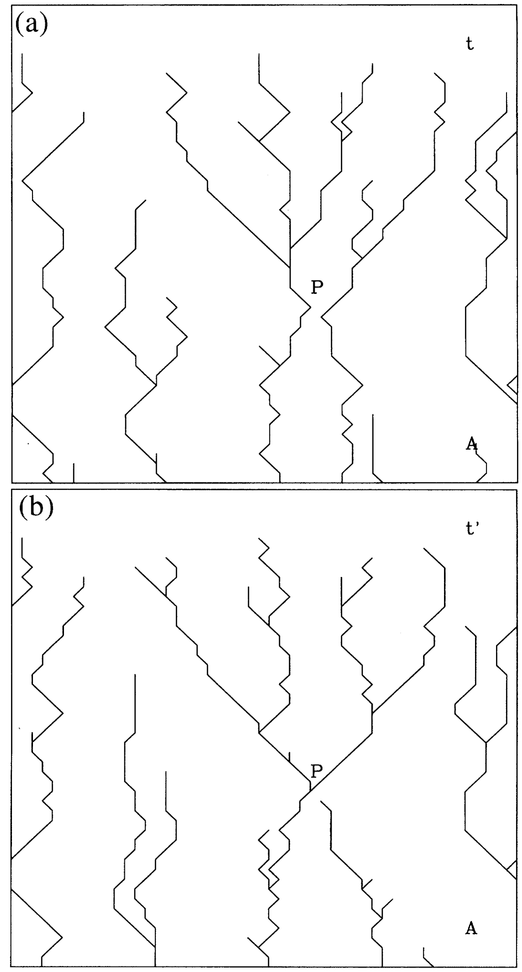

FIG. 4. A snapshot of a typical river network created by our dynamics at two successive times t (a) and $t' > t$ (b). Only sites with $s \geq 30$ are shown. The values of t and t' correspond to 89 980 and 90 000 temporal iterations with $\Delta t = 10^{-4}, \Delta x = 0.5, M = 1$, and $D_2/D_1 = -0.1$. The initial condition was a flat configuration. The flow is constructed as follows: Starting from a site (x, y) the water can flow only to one of the three sites $(x + 1, y + 1), (x, y + 1)$, or $(x - 1, y + 1)$, thus causing an overall flow downward (from y to $y + 1$). The choice of which of the three sites is determined by the relative heights of the corresponding h's (steepest descent). In the region indicated by P one main stream, which was present in (a), has been captured by a neighboring main stream in (b) (piracy effect). An example of a stream present in (a) which has disappeared in (b) (abstraction effect) is also shown in the region A. The boundary conditions employed were periodic in the x direction (perpendicular to the flow) and free in the y direction (parallel to the flow).

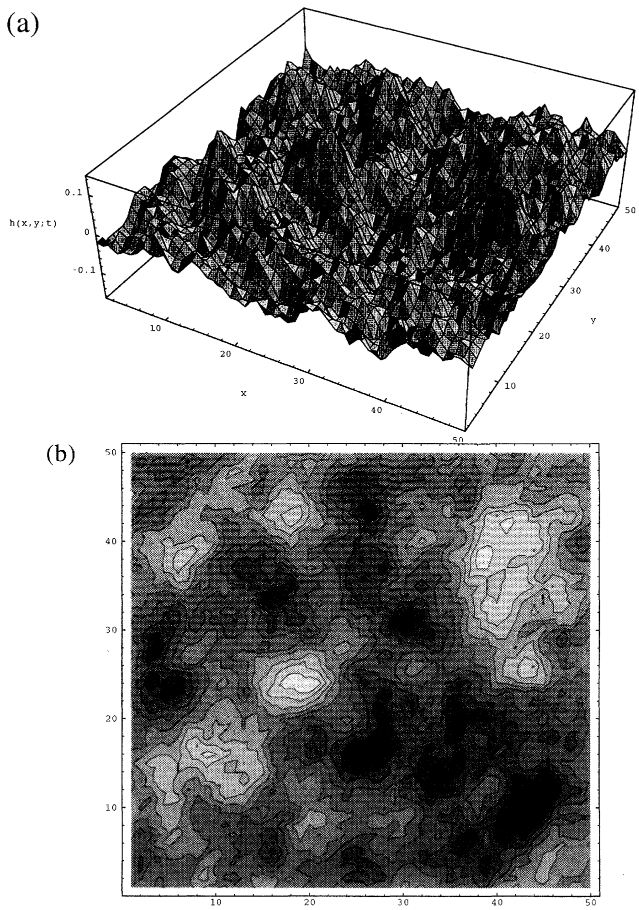

FIG. 5. Landscape (a) and contour plot (b) at t' corresponding to Fig. 4(b). Values for the parameters are the same as Fig. 4(b)

only sites with $s \geq 30$ are shown, while the corresponding elevation profile is shown in Fig. 5. The model captures some of the key ingredients of Glock's theory of the evolution of river networks [5]—small tributaries are eliminated as the main rivers swell in size. The figures also show the effects of piracy in which a larger aggressive stream captures the flow of a neighboring stream. Such effects had also been shown to occur by Leheny and Nagel [10] in their lattice model for river networks.

A more realistic model than the one presented here would have coupled differential equations for the variables $h(\mathbf{x}, t)$ and the flow s. Such a coupling would capture nonlocal effects present in real basins.

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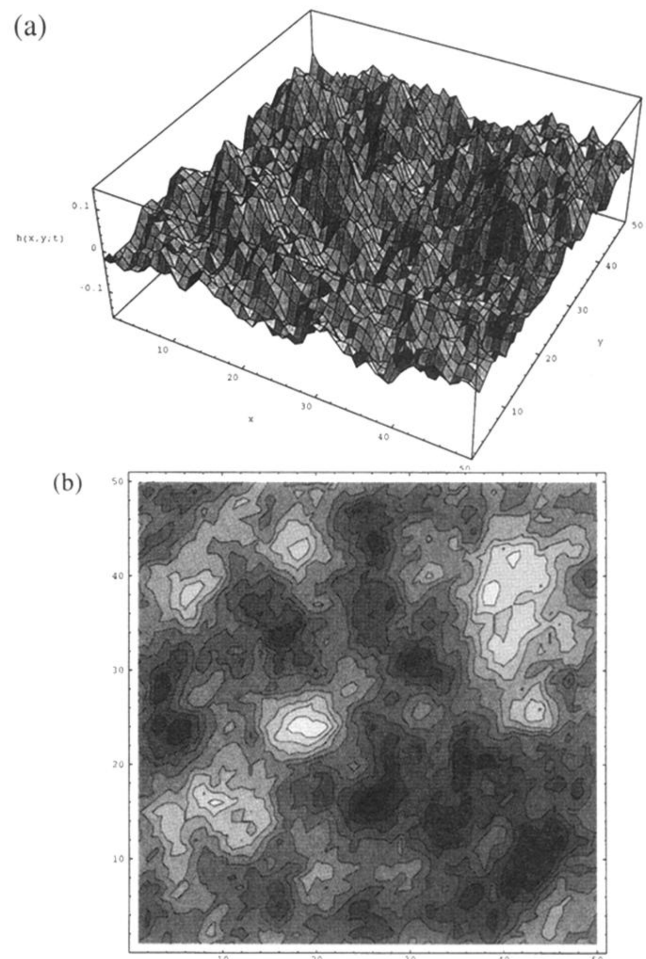

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