ELSEVIER

Physica A 224 (1996) 232-238

PHYSICA A

Multifractal scaling in Sinai diffusion

K.P.N. Murthy [a], A. Giacometti [b], K.W. Kehr [b]

[a] Theoretical Studies Section, Materials Science Division, Indira Gandhi Center for Atomic Research, Kalpakkam 603 102, Tamil Nadu, India

[b] Institut für Festkörperforschung des Forschungszentrums Jülich, Postfach 1913, D-52425, Jülich, Germany

Abstract

We consider the mean first passage time of random walks to go from one end of a segment of a Sinai lattice to the other. We show that the distribution of the mean first passage time over Sinai disorder obeys a finite-size-multiscaling form.

1. Introduction

The transport characteristics of disordered media are different from those of regular (homogeneous or periodic) systems, see [1]. The diffusion process is often anomalous. No general theory of anomalous diffusion has yet been formulated and one focusses attention on simple prototypes. The Sinai model [2] proposed in the early eighties is perhaps the simplest of disorder models that exhibit anomalous diffusion.

Sinai considered a particle executing a random walk on a one-dimensional lattice. When at lattice site j, the particle can hop to $j \pm 1$ with probabilities p_j and $1 - p_j$ respectively. Disorder is modelled by prescribing $\{p_j\}$ to be a set of independent and identically distributed random variables. p_j is bounded away from 0 and 1, and the logarithm of $(1-p_j)/p_j$ has zero mean and finite variance, σ^2. Notice that this condition implies that $\beta = \langle (1 - p_j)/p_j \rangle$ is greater than unity, where the angular brackets denote an average over the distribution of p_j. For typical disorder configurations, Sinai showed [2] that the mean square displacement increases very slowly as the fourth power of the logarithm of time.

An alternate description of the random walk process consists of investigating the statistics of the first passage time (FPT) to a given site N (absorbing right boundary), starting from an initial site 0 (reflecting left boundary). At site 0, we have p_0 as the right jump probability and $1 - p_0$ as the sojourn probability. Let $t_{0,N}$ denote the FPT.

0378-4371/96/$15.00
SSDI 0378-4371(95)00323-1

For a given realization of the set $\{p_i; i = 0, N-1\}$, let $t = \bar{t}_{0,N}$ be the mean first passage time (MFPT), where overbar denotes average over an ensemble of all possible random walks that start at 0, and eventually terminate at N. An exact expression for t in terms of the quenched jump probabilities can be derived, see [3], and is given below:

$$t = \sum_{k=0}^{N-1} \frac{1}{p_k} + \sum_{k=0}^{N-2} \frac{1}{p_k} \sum_{i=k+1}^{N-1} \prod_{j=k+1}^{i} \frac{1-p_j}{p_j}. \tag{1}$$

Disorder average of t can be carried out explicitly and asymptotically ($N \to \infty$) we get

$$\langle t \rangle \sim \beta^N, \tag{2}$$

where the angular brackets denote an average over $\{p_i; i = 0, N-1\}$. We observe that this result corresponds to that of simple random walks with $p_j = p < \frac{1}{2}$, $\forall j$. The bias $(1-p)/p$ is disorder dependent and equals β. A relevant statistics in the context of disordered systems is the typical value, denoted by $\hat{t}$ and is defined as $\exp[\langle \ln t \rangle]$. Noskowicz and Goldhirsch [4] showed that asymptotically

$$\hat{t} \sim \exp\left[\sigma\sqrt{\pi/2}N^{1/2}\right], \tag{3}$$

which is much smaller than $\langle t \rangle$, see also [3,5]. This suggests that the distribution of MFPT over the Sinai disorder is broad with a long tail.

For the purpose of numerical investigations, it proves useful to consider a simple dichotomic model of the Sinai disorder, and accordingly we prescribe p_j to take values of $\frac{1}{2} \pm \epsilon$ with equal probabilities. The parameter ϵ measures the strength of disorder, and can take values between 0 and $\frac{1}{2}$. The larger the value of ϵ, the stronger is the disorder. $\epsilon = 0$ corresponds to simple random walks, with $p_j = \frac{1}{2}$, $\forall j$. It is easily verified that the above prescription for the distribution of p_j obeys the Sinai condition. A Monte Carlo simulation [6], employing the dichotomic model, indicated that the distribution of MFPT has an $1/t$ tail, see also [7]. A numerical study, employing exact enumeration technique, showed that the distribution of MFPT is multifractal [8], and the fractal dimension varies with ϵ. In a recent study [9] we derived an analytical expression for the variation of the fractal dimension with ϵ and demonstrated that the fluctuations of the MFPT from one disorder configuration to the other exhibit statistical self-similarity. We established explicitly, see [9], the connection between the generalized Renyi dimension [10] characterizing the multifractal behaviour and the exponents $\xi(q)$ (defined below) characterizing the divergence of the moments of the distribution of MFPT.

In this paper we shall show the connection between the aforementioned characterizations of multifractal behaviour and the finite-size-multiscaling description introduced in Ref. [11]. We demonstrate *explicitly* that at least within the framework of this simple toy model, all the three seemingly different characterizations of multifractal behaviour are related, each to the other.

2. Distribution of MFPT

Consider the dichotomic model of the Sinai disorder. For a given N, there are 2^N possible disorder configurations. The MFPT for each configuration can be calculated exactly, employing Eq. (1) and let $\{t_i; i = 1, 2^N\}$ denote these values. It is seen that $\tau = \min\{t_i\}$ is obtained when $p_j = \frac{1}{2} + \epsilon$, $\forall j$. From Eq. (1) we get

$$\tau = \frac{N}{2\epsilon} + \frac{1-2\epsilon}{8\epsilon^2}\left[\left(\frac{1-2\epsilon}{1+2\epsilon}\right)^N - 1\right], \tag{4}$$

which diverges linearly, for large N. On the other hand, $T = \max\{t_i\}$, obtained when $p_j = \frac{1}{2} - \epsilon$, $\forall j$, and is given by

$$T = -\frac{N}{2\epsilon} + \frac{1+2\epsilon}{8\epsilon^2}\left[\left(\frac{1+2\epsilon}{1-2\epsilon}\right)^N - 1\right]. \tag{5}$$

T diverges exponentially with N. Thus the distribution of the MFPT defined between $(\sim N)$ and $(\sim e^N)$, becomes broader and broader as $N \to \infty$. Fig. 1 depicts log–log plots of the distributions for $\epsilon = 0.35$ and N ranging from 10 to 26 in steps of 2. Our aim here is to investigate the asymptotic scaling form of the MFPT distribution and to this end we turn our attention below.

3. Scaling of the MFPT distribution

Let $\rho(t,T)$ denote the distribution the MFPT, where we have chosen T, see Eq. (5), as the system parameter for characterising the scaling behaviour. This is a natural choice. The random variable t is *time* rather than *length*, and hence in a sense T takes the same role as the system size in finite-size-scaling pictures [12]. This analogy however, should be taken with caution since T is related to the system size (N in our terminology) by Eq. (5).

In a simple finite-size-scaling approach we would say that the distribution ρ scales as

$$\rho(t,T) = t^{-\nu} g(t/T^{\mu}) = T^{-\gamma}\tilde{g}(t/T^{\mu}), \tag{6}$$

where $\nu = \gamma/\mu$ and $\tilde{g}(x) = g(x)/x^{\nu}$. In the above, ν and μ are the scaling exponents and $g(x)$ the scaling function, such that $g(x) \to$ a constant as $x \to 0$, thus ensuring that one does indeed get a power law $t^{-\nu}$ as $T \to \infty$, and $g(x) \to 0$ (sufficiently fast), in order to have a well-behaved distribution at infinity. By a simple change of variables it is easy to show that the distribution (6) is normalized to unity only if $\nu > 1$ (and in that case $g(0) = \nu - 1$).

On the other hand if we consider the moments of t, they behave, see [9], as

$$M(q,T) = \int dt\, t^q \rho(t,T) \sim T^{\xi(q)}, \tag{7}$$

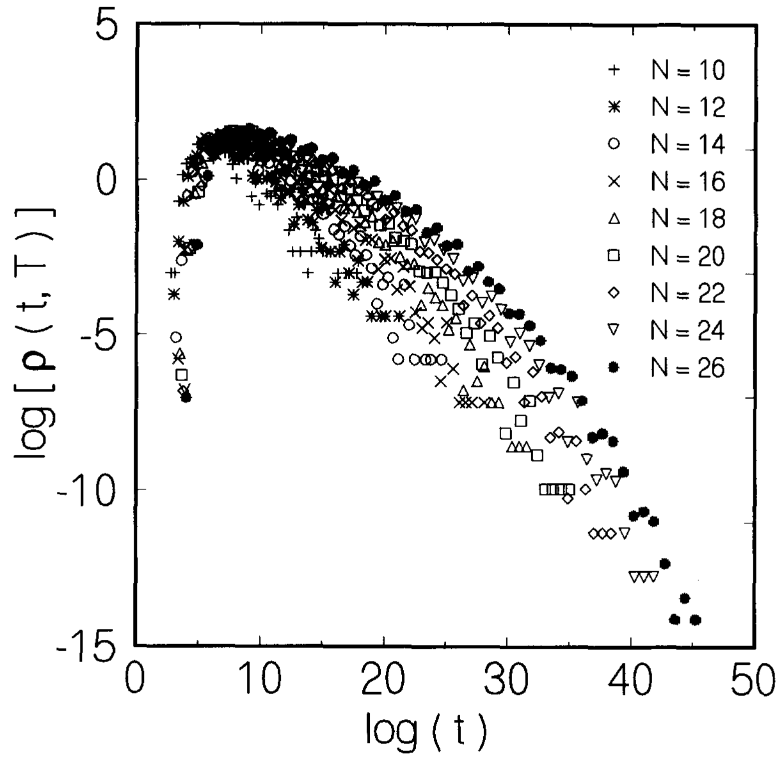

Fig. 1. Distribution of MFPT; $\epsilon = 0.35$.

where $\xi(q)$ is the exponent characterising the asymptotic ($T \to \infty$) behaviour of the qth moment. Assuming (6) one can easily derive that $\xi(q) = \mu(q+1-\nu)$ if $q > \nu - 1$ and $\xi(q) = 0$ otherwise. In this simple case a plot of $t^{\nu}\rho(t,T)$ versus t/T^{μ} would render the different data sets corresponding to different values of T (or equivalently N) collapse to a single scaling curve. We tried this scaling with several pairs of values of (ν, μ) for $\epsilon = 0.35$ and N ranging from 10 to 26 in steps of 2. We could not detect any tendency for data sets to collapse.

The generalization of the above picture is a *multi-finite-size-scaling* ansatz for a family of functions $\rho(t,T)$ that depend on the parameter α, see [11], and is given below:

$$\rho(t,T) = t^{-\psi(\alpha)} g\left(\frac{t}{T^{\alpha}}\right) = T^{-f(\alpha)} \tilde{g}\left(\frac{t}{T^{\alpha}}\right). \tag{8}$$

In the above, as before, $\psi(\alpha) = f(\alpha)/\alpha$ and $\tilde{g}(x) = g(x)/x^{\psi(\alpha)}$.

The simple scaling and multifractal scaling would agree with each other only for the case when there is a single scaling function and $\xi(q)$ is linear in q. Otherwise there would exist a continuous spectrum of scaling exponents, α and $f(\alpha)$ corresponding to all the values taken on by $q = df/d\alpha$.

In practice, to detect if the data sets obey an asymptotic multifractal scaling form, we should plot $\log[\rho(t,T)]/\log(T)$ versus $\log(t)/\log(T)$ for several large values of T.

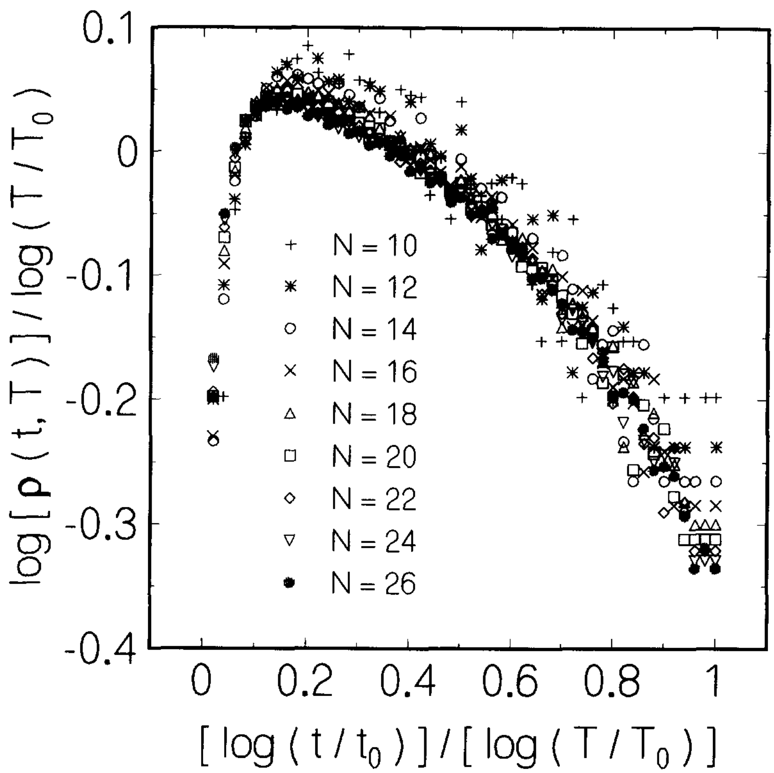

Fig. 2. Multifractal scaling form of the distribution of MFPT. The scaling parameter T is a function of N and ϵ, see Eq. (5); $\epsilon = 0.35$; $t_0 = T_0 = \tau$; see Eq. (4), for an expression of τ in terms of N and ϵ.

This is called f–α representation of the data sets. Fig. 2 depicts nine sets of data in the f–α representation for the case with $\epsilon = 0.35$ and N ranging from 10 to 26 in steps of 2. We observe a good collapse of the nine data sets. In our analysis, following [11], we have expressed t and T in units of τ, and hence in Fig. 2, $t_0 = T_0 = \tau$, see Eq. (4) for an expression of τ in terms of N, and ϵ. We observe that in Fig. 2, if we delete the data points that correspond to $N = 10$, 12 and 14, the remaining six data sets collapse more sharply. This indicates that the asymptotic multifractal scaling form is perhaps reached when N is greater than 16 or so.

A final remark is in order [1]. Following [13], we can define a *potential*

$$U(i) = \sum_{j=1}^{i-1} \log \frac{(1-p_j)}{p_j}. \tag{9}$$

Then from Eq. (1) it is easy to see [3–5] that for $N \gg 1$,

$$\frac{\log t}{N^{1/2}} \overset{N\to\infty}{\longrightarrow} x, \tag{10}$$

[1] We are indebted to Deepak Dhar for bringing this point to our attention.

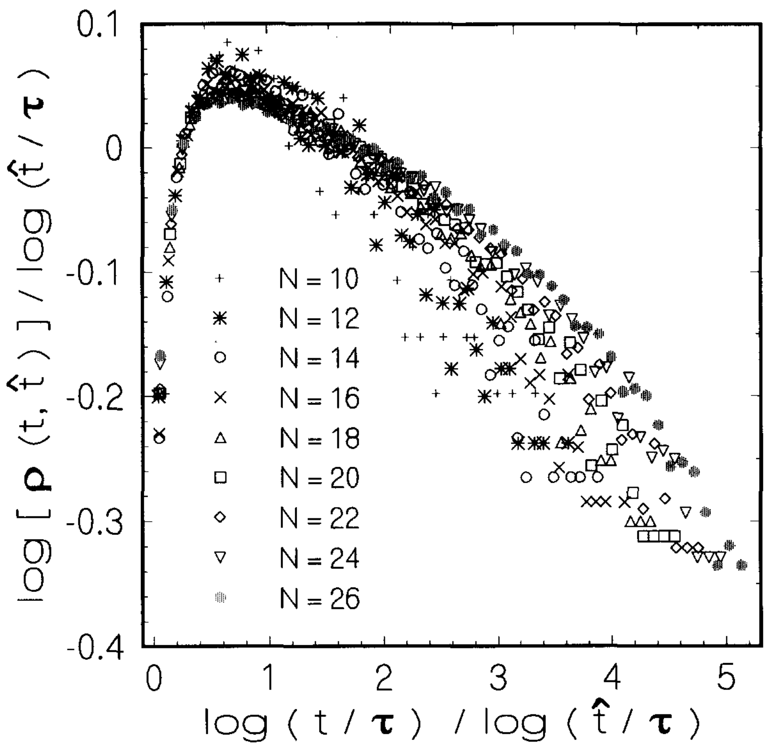

Fig. 3. Same as in Fig. 2 but with T replaced by $\hat{t}$, the typical MFPT.

where x is the appropriate continuum limit of the above quantity. According to this argument, it appears that the correct scaling variable to be used in Fig. 2 is $\log(t)/\log(\hat{t})$ and not $\log(t)/\log(T)$. We have checked this out, and Fig. 3 depicts the data sets in the f–α representation with $\hat{t}$ taken as the scaling parameter. The collapse of the data sets is much less satisfactory than in the previous case. We feel that the reason for this discrepancy should be traced to the presence of rare events with high values of t which are precisely the ones responsible for the fact that typical and mean values of MFPT do not coincide. We plan to pursue these investigations by taking N far beyond 26 and for different values of ϵ in order to clarify this issue. Also, we plan to investigate whether there also exists a scaling form with respect to the strength of disorder, ϵ.

4. Conclusions

We have investigated the distribution of the mean first passage time over Sinai disorder. We find that the distribution has a finite-size-multiscaling form. Our present results provide with the missing step in the connections amongst the three possible characterizations of multifractality within the framework of the model.

Acknowledgements

One of the authors (KPNM) is thankful to M.C. Valsakumar, R. Indira, V. Sridhar and S. Kanmani for useful discussions. We thank Deepak Dhar for a critical reading of the manuscript.

References

[1] J. Haus and K.W. Kehr, Phys. Rep. 150 (1987) 263;
S. Alexander, J. Bernasconi, W.R. Schneider and R. Orbach, Rev. Mod. Phys. 53 (1981) 175;
S. Havlin and D. ben-Avraham, Adv. Phys. 36 (1987) 695;
J.P. Bouchaud and A. Georges, Phys. Rep. 195 (1990) 127.

[2] Ya G. Sinai, Theory Prob. Appl. 27 (1982) 247; Mathematical problems in theoretical physics, in: Lectures Notes in Physics 153, eds. J. Ehlers et al. (Springer, Berlin, 1982).

[3] K.P.N. Murthy and K.W. Kehr, Phys. Rev. A 40 (1989) 2082; erratum 41 (1990) 1160.

[4] S.H. Noskowicz and I. Goldhirsch, Phys. Rev. Lett. 61 (1989) 500.

[5] P. Le Doussal, Phys. Rev. Lett. 62 (1989) 3097.

[6] K.W. Kehr and K.P.N. Murthy, Phys. Rev. A 41 (1990) 5728.

[7] S.H. Noskowicz and I. Goldhirsch, Phys. Rev. A 42 (1990) 2047.

[8] K.P.N. Murthy, S. Rajasekar and K.W. Kehr, J. Phys. A 27 (1994) L107.

[9] K.P.N. Murthy, K.W. Kehr and A. Giacometti, Multifractal scaling of moments of mean first passage time in the presence of Sinai disorder, Phys. Rev. E (1995), submitted.

[10] J. Mandelbrot, J. Fluid. Mech. 62 (1974) 331;
U. Frisch and G. Parisi, in: Turbulence and Predictability of Geophysical Flows and Climate Dynamics, Proc. Int. Sch. Phys. "Enrico Fermi", eds. M. Ghil, R. Benzi and G. Parisi (North-Holland, New York, 1985) p. 84;
T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia and B. Shraiman, Phys. Rev. B 33 (1986) 1141.

[11] L.P. Kadanoff, S.R. Nagel, Lei Wu and Su-min Zhou, Phys. Rev. A 39 (1989) 6524.

[12] See e.g. M. Fisher, in: Critical Phenomena, Proc. 51st Enrico Fermi Summer School, Varenna, ed. M.S. Green (Academic Press, New York, 1972).

[13] E. Marinari, G. Parisi, D. Ruelle and P. Windey, Phys. Rev. Lett. 50 (1983) 1223.