

Macroeconomics (Cazzavillan)

Handout 2

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1 Stability of homogeneous ordinary differential equations

Proposition 1. The linear differential equation $\dot{x}(t) = ax(t)$ is locally asymptotically stable if $a < 0$. Moreover, the nonlinear equation $\dot{x}(t) = g[x(t)]$, where g is such that it is continuous and differentiable at x^* with $g(x^*) = 0$, then $x(t)$ is locally stable if $\partial g[x(t)]/\partial t < 0$.

Heuristic proof.

$$\dot{x}(t) = g[x(t)]$$

$$g(x^*) = 0$$

If g is upward sloping $\partial g(t)/\partial t > 0$, $\Rightarrow x_1 > x/2 \Rightarrow g(x_1) > g(x_2)$.

Thus, if $x(t) < x^*$,

$$g[x(t)] < g[x^*]$$

$$\dot{x}(t) < 0$$

If $x(t) > x^*$,

$$g[x(t)] > g[x^*]$$

$$\dot{x}(t) > 0$$

Intuition: when x is small, it falls even further away from the initial value. When it is too big, it grows even more, always in the opposite direction of the initial value. These behavior suggest instability and a tendency to move away from the initial equilibrium.

If g is downward sloping $\partial g(t)/\partial t < 0$, $\Rightarrow x_1 > x/2 \Rightarrow g(x_1) < g(x_2)$.

Thus, if $x(t) < x^*$,

$$g[x(t)] > g[x^*]$$

$$\dot{x}(t) > 0$$

If $x(t) > x^*$,

$$g[x(t)] < g[x^*]$$

$$\dot{x}(t) < 0$$

Intuition: when x is small, it grows whereas when it is too big it tends to fall. Therefore, for some initial values, there is a tendency to converge to a certain value.

2 Stability of a system of homogeneous ordinary differential equations

The most straightforward way to study such systems is to start with diagonal systems like:

$$\dot{y}(t) = a_{11}y(t)$$

$$\dot{x}(t) = a_{22}x(t)$$

Extending the analysis done above, the system is stable if both coefficients are negative whereas it is unstable if they are positive. If they are opposite sign we are in a situation where there is a stable arm, conducive to the steady state and associated with the negative coefficient. There is also an unstable arm associated with the positive coefficient that bring us away from the steady state. Consider all cases: I will do that in words, try to make the corresponding graph to check you understood. The locus $\dot{x}(t) = 0$ is the y -axis whereas the locus $\dot{y}(t) = 0$ is the x -axis.

Case 1: $a_{11} < 0$ and $a_{22} < 0$

If we increase y , $\dot{y}(t) < 0$; in the same way if we increase x , $\dot{x}(t) < 0$. Therefore you can check graphically that no matter where you start, you will always be brought back to the steady state, in this case the point $(0,0)$.

Case 2: $a_{11} > 0$ and $a_{22} > 0$

If we increase y , $\dot{y}(t) > 0$; in the same way if we increase x , $\dot{x}(t) > 0$. Therefore you can check graphically that no matter where you start you tend to escape away from the steady state.

Case 3: $a_{11} < 0$ and $a_{22} > 0$

If we increase y , $\dot{y}(t) < 0$; instead, if we increase x , $\dot{x}(t) > 0$. In this case the x -axis is an explosive path whereas the y -axis is the saddle path. Therefore if you start on the saddle path you converge to the steady state, otherwise not.

To study the dynamics in a diagonal system it is sufficient to look at the sign of the elements in the diagonal. But what if we do not have a diagonal system? Non-diagonal systems (associated with the matrix A) can be diagonalized using the matrix of the eigenvectors (V) to obtain a new diagonal system, whose associated matrix is the diagonal matrix of the eigenvalues (D). After this transformation, the dynamics of the systems can be studied by looking at the sign of the diagonal elements of such matrix, namely the eigenvalues. Moreover, the determinant of a matrix is equal to the product of the eigenvalues and therefore, in the 2×2 case, by looking at the sign of the determinant we can infer the sign of the eigenvalues without computing them.