Macroeconomics (Cazzavilan)
Handout 1

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1 Where does the Hamiltonian come from? An heuristic derivation solving the Ramsey problem

Assume that the economic agent can choose the path of consumption, c(t), and capital, k(t), so as to maximize:

$$
\begin{aligned}
U(0) &= \int_0^T exp^{-\rho * t} u(c(t))dt \\
s.t.\dot{k}(t) &= f(k(t)) - c(t) - \delta * k(t) \\
k(0) &> 0 \\
k(T)exp^{-\overline{r}(T)T} &\geq 0
\end{aligned}
$$

where the integral is from 0 to T. The last condition is also called the trasversality condition. It is a complementary-slackness condition that says that the stock of capital at the end of the period cannot be negative: either it is zero with a positive price, or if it is positive it has to have a zero value.

As first step let us apply the Khun-Ticker Theorem for solving static optimization problems. The Lagrangian will take the form:

$$
L = \int_0^T exp^{-\rho * t} u(c(t))dt + \int_0^T \mu(t)[f(k(t)) - c(t) - \delta * k(t) - \dot{k}(t)]dt + \nu * k(T)exp^{-\overline{r}(T)T}
$$

$$
= \int_0^T exp^{-\rho * t} u(c(t))dt + \int_0^T \mu(t)f(k(t))dt
$$

$$
- \int_0^T \mu(t)c(t)dt - \int_0^T \mu(t)\delta * k(t)dt - \int_0^T \mu(t)\dot{k}(t)dt + \nu * k(T)exp^{-\overline{r}(T)T}
$$

Next step would be to derive the first order conditions with respect to $c(t)$ and $k(t)$. However, we do not know how to take the derivative of $\mu(t)\dot{(k(t))}$.

Therefore, using integration by part with substitute this term with something we can take the derivative of.

$$\int_0^T \mu(t)\dot{k}(t)dt = \mu(t)k(t)dt + \int_0^T \dot{\mu}(t)k(t)dt$$

Taking the derivative with respect to time

$$\mu(t)\dot{k}(t)dt = \frac{\partial \mu(t)k(t)}{\partial t} + \dot{\mu}(t)k(t)$$

Now integrating between 0 and T

$$\int_0^T \mu(t)\dot{k}(t)dt = \mu(T)k(T) - \mu(0)k(0) - \int_0^T \dot{\mu}(t)k(t)dt$$

Now we replace in the Lagrangian $\int_0^T \mu(t)\dot{k}(t)dt$ with $\mu(T)k(T) - \mu(0)k(0) - \int_0^T \dot{\mu}(t)k(t)dt$

$$L = \int_0^T exp^{-\rho * t}u(c(t))dt + \int_0^T \mu(t)f(k(t))dt - \int_0^T \mu(t)c(t)dt - \int_0^T \mu(t)\delta * k(t)dt - \mu(T)k(T) + \mu(0)k(0) + \int_0^T \dot{\mu}(t)k(t)dt + \nu * k(T)exp^{-\overline{r}(T)T}$$

$$L = \int_0^T [exp^{-\rho * t}u(c(t)) + \mu(t)[f(k(t)) - c(t) - \delta k(t)]]dt - \mu(T)k(T) + \mu(0)k(0) + \int_0^T \dot{\mu}(t)k(t)dt + \nu * k(T)exp^{-\overline{r}(T)T}$$

The expression within the first integral is the so called Hamiltonian:

$$H(k,c,t,\mu) = exp^{-\rho * t}u(c(t)) + \mu(t)[f(k(t)) - c(t) - \delta k(t)]$$

$$L = \int_0^T [H(k,c,t,\mu) + \dot{\mu}(t)k(t)]dt - \mu(T)k(T) + \mu(0)k(0) + \int_0^T \dot{\mu}(t)k(t)dt + \nu * k(T)exp^{-\overline{r}(T)T}$$

Let $c\tilde{(t)}$ and $k\tilde{(t)}$ be the optimal path for the control and state variable and consider a small perturbation around the optimal path:

$$c(t) = \widetilde{c(t)} + \epsilon p1(t)$$

$$k(t) = \widetilde{k(t)} + \epsilon p2(t)$$

$$k(T) = \widetilde{k(T)} + \epsilon dk(T)$$

If the initial path is optimal, then $\partial L/\partial \epsilon$ should be equal to zero.

Substituting c(t) and k(t) with the perturbation, the Lagrangian can be rewritten as a function of ϵ:

$$L = \int_0^T [H(k(..\epsilon), c(..\epsilon) + \dot{\mu}(..)k(..\epsilon)]dt - \mu(T)k(T,\epsilon) + \mu(0)k(0) + \nu * k(T,\epsilon)exp^{-\overline{r}(T)T}$$

$$\partial L/\partial \epsilon = 0$$

Considering that:

$$\partial H(k(..\epsilon, c(..\epsilon)/\partial \epsilon = \partial H(k(..\epsilon, c(..\epsilon)/\partial c * p1(t) + \partial H(k(..\epsilon, c(..\epsilon)/\partial k * p2(t)$$

and

$$\dot{\mu}\partial k/\partial \epsilon = \dot{\mu}p2(t)$$

and

$$\partial k(T)/\partial \epsilon = dk(T)$$

and

$$\mu(T)\partial k(T)/\partial \epsilon$$

$$\partial L/\partial \epsilon = \int_0^T \partial H/\partial c * p1(t) + [\partial H/\partial k + \dot{\mu}] * p2(t)dt$$
$$+[\nu exp^{-\overline{r}(T)T)} - \mu(T)]dk(T)$$

$$\partial L/\partial \epsilon =>$$

$$\partial H/\partial c = 0$$
$$\partial H/\partial k + \dot{\mu} = 0$$
$$\nu * k(T, \epsilon)exp^{-\overline{r}(T)T} = \mu$$

Using the last condition the trasversality condition can be re-written:

$$\mu(T)k(T) = 0$$

In the case of infinite planning horizon, the first-order conditions are the same. The only difference is the trasversality condition, that is now defined for the limit of t going to infinity.

$$lim_{n\rightarrow\infty}\mu(T)k(T) = 0$$