Analytical Solutions for Circular Stratified Eddies of the Reduced-Gravity Shallow-Water Equations

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ABSTRACT

A set of new analytical nonstationary solutions of the nonlinear, reduced-gravity shallow-water equations on an f plane in a vertically stably stratified active environment is presented. The solutions, which describe the dynamics of inertially pulsating surface as well as intermediate lens-like stratified vortices, represent an extension of previous analytical solutions to more realistic vortex shapes and structures of the vortex swirl velocity fields in the presence of an arbitrary stable vertical stratification within the active environment. To elucidate aspects of the novelty of the new set of solutions, examples are presented referring to a vertically stratified surface vortex and to a vertically stratified intermediate vortex, both characterized by azimuthal velocity fields that are nonlinear functions of the radius and by layer shapes that largely deviate from paraboloidal shapes. First, a solution describing a five-layer surface “warm core” eddy is analyzed and characteristics resembling characteristics observed in geophysical surface vortices are revealed: Within each layer, the obtained azimuthal velocity shows a realistic distribution, as its maximum is located far from the vortex rim, where it is negligible. Moreover, the obtained azimuthal velocity is largest in the surface layer and decreases monotonically toward the deeper layers. Integral properties of the new stratified solutions significantly differ from the corresponding properties of equivalent, known homogeneous solutions. Significant differences are found, for instance, between the shape of a five-layer vortex and that of its homogeneous counterpart having the same mean azimuthal velocity structure. Second, a solution referring to a five-layer intermediate “meddy like” vortex is analyzed: while in each layer the obtained azimuthal velocity is maximum in the interior part of the vortex and decreases toward its center and its periphery, its magnitude is largest in the intermediate layer and it decreases toward the surface and toward the bottom layer, which are characteristics resembling characteristics of observed meddies. The quoted examples demonstrate that the new solutions add substantial realism to the analytical description of oceanic nonlinear geophysical vortices. In the context of the nonlinear reduced-gravity shallow-water equations on an f plane they seem to represent the most general analytical features achievable, which refer to vertically stratified circular vortices characterized by linear radial velocity fields.

1. Introduction

Much attention has been devoted, in the last decades, to the investigation of lens-like, surface, as well as intermediate oceanic geophysical vortices (see, e.g., Saunders 1971; Armi and Zenk 1984; Cushman-Roisin et al. 1985; Cushman-Roisin 1987; Käse and Zenk 1987; Pavia and Cushman-Roisin 1988; Rogers 1989; Olson 1991; Matsuura 1995; Cushman-Roisin and Merchant-Both 1995; Ochoa et al. 1998; Rubino et al. 1998, 2002, 2003; Dotsenko et al. 2004; Gascard et al. 2002; Rubino and Brandt 2003; Dengler et al. 2004; Rubino and Dotsenko 2006). In this frame, a conspicuous number of observational, experimental, and theoretical studies have revealed the ubiquity and, often, the remarkable longevity of geophysical lens-like vortices and have demonstrated their significant role in the larger-scale oceanic circulation, exerted, for example, in their transfer of anomalous water mass characteristics across frontal regions and in their influence on oceanic mixing as well as on deep penetrating oceanic convection (see,
Often, the theoretical investigation of lens-like geophysical vortices has been based on the nonlinear, shallow-water reduced-gravity equations in a rotating frame (see, e.g., Csanady 1979; Nof 1983; Killworth 1983; Cushman-Roisin et al. 1985; Cushman-Roisin 1987; Olson 1991; Rubino et al. 1998, 2002; Rubino and Dotsenko 2006). These equations refer to a simplified dynamics (e.g., the “exterior” ocean is supposed motionless and baroclinic instabilities are not allowed); nevertheless their usefulness for understanding many bulk characteristics of vortical dynamics has been demonstrated.

In particular, one remarkable peculiarity of the nonlinear, reduced-gravity shallow-water equations on an f plane is that they can be solved analytically in special cases of geophysical interest (see, e.g., Ball 1963; Thacker 1981; Young 1986; Cushman-Roisin 1987; Rogers 1989; Rubino et al. 1998; Rubino and Dotsenko 2006). Such a characteristic has a particular relevance in oceanic sciences: apart from their intrinsic value and fascinations, analytical solutions of nonlinear equations referring to the dynamics of geophysical motions can help to evidence fundamental aspects inherent in the dynamics of observed flow features that may be difficult to unravel when their whole complexity is addressed (Esenkov and Cushman-Roisin 1999; Rubino and Dotsenko 2006). Moreover, such solutions can be used for assessing the performance of numerical models (see, e.g., Sun et al. 1993; Esenkov and Cushman-Roisin 1999; Rubino et al. 2002). It is thus not surprising that, in the last decades, analytical solutions of the nonlinear reduced-gravity shallow-water equations referring to geophysical rotating vortices have been presented in different investigations. Cushman-Roisin (1987) found a solution describing the dynamics of an elliptical geophysical surface lens-like vortex that rotates steadily without deformation (the “rodon”). This solution was further discussed by Cushman-Roisin et al. (1985), who found that three fundamental oscillation modes are possible for such elliptical geophysical surface vortices on an f plane. Extensions of the rodon solution to nonstationary surface vortices (“pulsons”) were presented, for example, by Young (1986), Cushman-Roisin (1987), and Rogers (1989). In all these solutions, which refer to homogeneous oceanic features, the obtained vortices are characterized by parabolic sections and horizontal velocity components that are linear functions of the horizontal coordinates. Rubino et al. (1998) extended the degree of realism of analytical geophysical vortices, as they presented a new set of solutions for circular surface vortices with more general shapes and structures of the azimuthal velocity. While in the pulson solution the tangential velocity is maximum at the vortex rim, the solutions of Rubino et al. (1998) comprise vortices whose tangential velocity is negligible there. The degree of realism inherent in these new solutions was investigated experimentally by Rubino and Brandt (2003), who found evidence that such kinds of vortices rather than pulson-like ones emerge by the sudden release of a cylinder of buoyant water in a rotating ocean. Still, the restriction to homogeneous vortices held. Actually, the density of observed geophysical lens-like vortices is in general a function of both the horizontal as well as the vertical coordinates (see, e.g., Joyce 1984; Armi et al. 1989; Gascard et al. 2002; Dengler et al. 2004). Rubino and Dotsenko (2006) were able to extend the pulson solution to the description of surface as well as intermediate vortices of arbitrary stable vertical stratifications. As a result, cyclonic as well as anticyclonic horizontal azimuthal velocities were found to coexist on different vortex layers within parts of an inertial period, while vertical distributions of the vortex mean azimuthal velocity resembling observed vertical velocity distributions of surface as well as intermediate lens-like vortices were obtained. Still, these solutions are restricted to vortices having linear velocity fields and paraboloidal shape.

In the present paper we propose a new set of solutions describing the evolution of circular, surface, as well as intermediate lens-like stratified geophysical vortices. The new solutions are no longer restricted to the description of vortices having linear azimuthal velocity fields and paraboloidal shapes. Moreover, they refer to arbitrary, stably stratified features. As a result, a larger degree of realism is achieved than was possible using previous analytical solutions. The obtained solutions comprise, as a subset, the pulson solution (Cushman-Roisin 1987; Rogers 1989), the set of solutions presented by Rubino et al. (1998), as well as those described by Rubino and Dotsenko (2006). In the context of the nonlinear reduced-gravity shallow-water equations on an f plane, they seem to represent the most general analytical features achievable, which refer to stably vertically stratified circular vortices characterized by linear radial velocity fields.

The paper is organized as follows: In section 2 the reduction of the problem to a system of ordinary differential equations in time is presented. The analytical solution of the problem is given in section 3, and, in section 4, particular solutions for stratified warm-core eddies characterized by realistic horizontal and vertical dimensions and density fields are discussed. In section 5 the new solutions are adapted to the description of intermediate lens-like vortices and a particular solution
referring to a five-layer, realistic meddy-like vortex is presented. In section 6 the solutions are discussed and conclusions are presented.

2. Reduction of the model to a system of ordinary differential equations

Under the assumption of circular symmetry, the \(n\)-layer nonlinear reduced-gravity shallow-water equations for a vertically stratified rotating system (Fig. 1) can be written in cylindrical coordinates as

\[
\frac{\partial u_j}{\partial t} + u_j \frac{\partial u_j}{\partial r} + \frac{u_j v_j}{r} - f v_j = F_j, \tag{1}
\]

\[
\frac{\partial v_j}{\partial t} + u_j \frac{\partial v_j}{\partial r} + u_j v_j + f u_j = 0, \quad \text{and} \tag{2}
\]

\[
\frac{\partial h_j}{\partial t} + \frac{1}{r} \frac{\partial (r u_j h_j)}{\partial r} = 0, \tag{3}
\]

where

\[
F_j = -g \sum_{k=1}^{j-1} (s_{k,j} - s_{k+1,j}) \frac{\partial h_k}{\partial r} - g \sum_{k=j}^{n} (1 - s_{k,n+1}) \frac{\partial h_k}{\partial r}, s_{k,j} = \frac{\rho_k}{\rho_j}.
\]

In the previous expressions, where the subscript \(j = 1, \ldots, n\) refers to the corresponding layer, \(r\) is the radial coordinate, \(t\) is the time, \(u_j(r, t)\) and \(v_j(r, t)\) are the radial and the azimuthal components of the horizontal velocity in the \(j\)th layer, and \(h_j(r, t)\) and \(\rho_j\) are the thickness and the density of \(j\)th (homogeneous) layer, \(g\) represents the acceleration due to gravity, and \(\rho_{n+1}\) indicates the density of the motionless surrounding fluid. Any solution of the system (1)–(3) describing surface, lens-like (frontal) vortices has to satisfy the conditions

\[
h_j(r_p, t) = 0, \quad j = 1, \ldots, n, \tag{4}
\]

where \(r = r_p\) represents the circular vortex outcropping line, which will be assumed here to be common to all layers in order to avoid discontinuities in the pressure terms at the vortex rim (see Fig. 1).

Let us now assume that the vortex thickness and velocity fields in the \(j\)th layer are characterized by the following horizontal structure (see, e.g., Rubino et al. 1998):

\[
u_j = A_j(t)r, \quad v_j = \sum_{i=1}^{N} B_{ij}(t)r^{2i-1} \quad \text{and} \tag{5}
\]

\[
h_j = \sum_{i=0}^{2N-1} C_{ij}(t)r^{2i}, \tag{6}
\]

where \(A_j, B_{ij}\), and \(C_{ij}\) are functions of time to be found. Note that, in the previous expressions (5) and (6), the number \(N \geq 1\) is the order of the solution.

The condition that the thickness of each layer is positive at \(r = 0\) leads to \(C_{ij0}(t) > 0\). Because the outcropping line \(r = r_p(t)\) is common to the different layers, \(n\) relations between the coefficients in (6) have to be satisfied:

\[
\sum_{i=0}^{2N-1} C_{ij}(t)r^{2i}(t) = 0, \quad j = 1, \ldots, n, \quad t \geq 0. \tag{7}
\]

Note that the thickness \(h_j\) of each layer must be positive for \(0 \leq r \leq r_p\).

Substitution of (5) and (6) into (1)–(3) yields a system of \((5N - 1) \times n\) coupled nonlinear ODEs and algebraic relations with respect to the unknown functions \(A_j, B_{ij}\), and \(C_{ij}\):

\[
\delta_{1j} \left( \frac{dA_j}{dt} + A_j^2 \right) - f B_{ij} - \sum_{m=1}^{i} B_{jm} B_{jm-i+1} + 2ig \left[ \sum_{k=1}^{i-1} (s_{k,j} - s_{k+1,j})C_{k,j} + \sum_{k=j}^{n} (1 - s_{k,n+1})C_{k,j} \right] = 0, \quad i = 1, \ldots, 2N - 1, \tag{8}
\]
\[ \frac{dB_{j,i}}{dt} + 2iA_jB_{j,i} + \delta_{j,i}fA_j = 0, \quad i = 1, \ldots, N, \]  
\[ \text{and} \]
\[ \frac{dC_{j,i}}{dt} + 2(i + 1)A_jC_{j,i} = 0, \quad i = 0, \ldots, 2N - 1, \]
where \( \delta_{j,i} \) is the Kronecker symbol \((\delta_{j,i} = 1, \delta_{j,i} = 0 \text{ for } j \neq i) \) and \( B_{j,i} = 0 \text{ for } i > N \). Note that, for \( 2 \leq i \leq 2N - 1 \) and for each \( j \), (8) represents a set of algebraic relations between the coefficients \( B_{j,i} (1 \leq i \leq N) \) and \( C_{j,i} (2 \leq i \leq 2N - 1) \).

3. Analytical solutions

Following Cushman-Roisin (1987) and Rubino et al. (1998), we search for periodic solutions of the system (8)–(10) having the following form:

\[ 2i g \left[ \sum_{k=1}^{i-1} (s_{k,i} - s_{k,n+1}) c_{k,i} + \sum_{k=j}^{n} (1 - s_{k,n+1}) c_{k,j} \right] = -\delta_l 1 \frac{1}{4} (1 - \gamma^2) f^2 + \sum_{m=1}^{i} l_{j,m} f, \]
where \( l_{j,i} = 0 \text{ for } i > N \).

Substitution of (13) into (7) leads to \( \sum_{i=0}^{2N-1} c_{j,i} r_i^2 = 0 \) for each \( j \). Introducing a new constant \( W \) such that \( r_b = W / \sqrt{W} \), the above equations can be rewritten as

\[ \sum_{i=0}^{2N-1} c_{j,i} W^{2i} = 0. \]

Note that \( W \) represents the radius of the outcropping line at the sea surface for \( t = t_0 \), where \( t_0 = -\varphi f \).

The system of equations in (14) and (15) can be solved analytically. To achieve this goal two different procedures can be used. In the first one all coefficients \( c_{j,i} \) satisfying (15) are considered as given. Thus, in this case, \( n \times (2N - 1) \) equations from (14) are given to find \( n \times N \) coefficients \( l_{j,i} \). The resulting system of nonlinear algebraic equations is thus overdetermined. In the second procedure all coefficients \( l_{j,i} \) are considered as given. Thus, in this case, \( 2N \times n \) equations are given to find \( 2N \times n \) unknown coefficients \( c_{j,i} \). In this paper we have used the second procedure, which seems simpler to us to be addressed. This has the physical meaning of prescribing the shapes of the vortex layers and to determine, on this basis, the vortex tangential velocity in each layer. We thus inserted the prescribed values of \( l_{j,i} \) in the system (14) and used the obtained values of \( c_{j,i} \) to evaluate \( c_{j,0} \) using the relation (15).

The exact analytical solution of (1)–(3) in the form (5) and (6) is

\[ u_j = \frac{\gamma f}{2} \frac{\cos(f t + \varphi)}{1 + \gamma \sin(f t + \varphi)} r_j, \quad j = 1, \ldots, n, \]
\[ v_j = \frac{-f}{2} \frac{r_j}{r_j + \sum_{i=1}^{N} \frac{l_{j,i}}{[1 + \gamma \sin(f t + \varphi)]^{i-1}}} r_i^{2i}, \]
and

\[ h_j = \sum_{i=0}^{2N-1} \frac{c_{j,i}}{[1 + \gamma \sin(f t + \varphi)]^{i+1}} r_i^2, \]
where \( l_{j,i} = 0 \text{ for } i > N \).

Note that to ensure the regularity of (16)–(18), formally the condition \(-1 < \gamma < 1\) has to be satisfied. This condition, however, is not sufficient to ensure the existence of positive \( h_j \) for \( 0 < r < r_b \) throughout the inertial period. Instead, \( \gamma \) has to belong to the narrower range \( |\gamma| < \gamma^* (\gamma^* < 1) \), where it can be shown [see Eq. (14)] that, in general, \( \gamma^* \) is a function of density distribution, free parameters \( l_{j,n} \), Coriolis parameter, and order of the solution. For a deeper discussion on the determination of \( \gamma^* \) in the case of the first-order, stratified plowon, the reader is referred to the work of Rubino and Dotsenko (2006).

The new analytical solutions given by (16)–(18) describe the temporal and spatial structure of nonlinearly
pulsating warm-core eddies. They represents an extension to the circular pulson solution of Cushman-Roisin (1987) and Rogers (1989), which can be recovered for \( N = 1 \) and \( n = 1 \) as well as to the stratified pulson of Rubino and Dotsenko (2006), which can be recovered for \( N = 1 \) and \( n \geq 1 \), as they can describe more general structures of the vortex shape and of its azimuthal velocity field. They represent also an extension to the solutions presented by Rubino et al. (1998), which can be recovered for \( N \geq 1 \) and \( n = 1 \), as they refer to an active multilayer ocean. Note, however, that the radial velocity is still a linear function of the radius and, as in the previous solutions quoted above, the velocity field is, in general, discontinuous at the vortex rim, where it drops to zero, as the surrounding ocean is assumed to be motionless. In the appendix to our paper, as an example, we present in detail a solution referring to the two-layer vortex of the second order \( (N = n = 2) \).

4. Description of a five-layer fourth-order surface vortex and comparison with an equivalent homogeneous fourth-order vortex

In general, in the case of high-order solutions \( (N > 1) \), shapes of the vortex interfaces as well as radial distributions of the azimuthal velocities are highly variable functions of \( \ell_j \). Thus, to obtain reasonably realistic looking eddies of higher order \( (N > 1) \), we first determined the values of \( \ell_{j,1} \) of first-order solutions \( (N = 1) \) yielding reasonably realistic looking eddies and then we introduced the further coefficients \( \ell_{j,i} (i > 1) \) as small deviations from \( \ell_{j,1} \) so that the positive thickness of all layers for \( 0 \leq r < r_{j}(t) \) throughout the inertial period was retained. To elucidate aspects of the novelty of the new set of solutions, let us consider a five-layer surface vortex \( (n = 5) \) in the case of a fourth-order solution \( (N = 4) \) for the following set of parameters: \( \rho_1 = 1025 \text{ kg m}^{-3}, \rho_j = \rho_1 + \delta(j - 1) \) for \( j = 2, \ldots, n, \rho_{n+1} = 1027 \text{ kg m}^{-3}, \delta = (\rho_{n+1} - \rho_1)/n, \gamma = 0.2, W = 10^5 \text{ m}, \text{ and } f = 0.7 \times 10^{-4} \text{ s}^{-1} \).

In Fig. 2 are depicted, as functions of the radial coordinate, the resulting azimuthal velocity and the corresponding position of the interface of each layer for a selected time within the inertial period, together with the corresponding quantities of the first-order solution (i.e., of the same solution in which the values of the higher-order coefficients are set to zero). In each layer, the maximum azimuthal velocity is not found at the eddy periphery (Fig. 2a), as it would be in the case of the pulson (Cushman-Roisin 1987). Moreover, the velocity is maximum in the surface layer and decreases toward the deeper layers. Note that both characteristics of the new solutions correspond to observed character-

![Fig. 2.](image)

FIG. 2. (a) Radial distribution of the vortex azimuthal velocity and (b) position of the vortex interfaces for a five-layer surface vortex at a selected time \( t_0 \) within the inertial period. Solid lines refer to the new solutions (fourth order), and dashed lines refer to the solution of the first order with linear azimuthal velocity and paraboloidal shape of the interfaces (Rubino and Dotsenko 2006).

In Fig. 3 are illustrated the inertial oscillations affecting the vortex total azimuthal transport \( V_j = \int_0^r h_j \rho_j \, dr \) along a radial section of each layer. As a consequence of the nonlinearity of the azimuthal velocity field and of the nonparaboloidal shapes of the interfaces, large variations in the amplitudes of the transport oscillations in the different layers are obtained. In the case discussed here, they are largest in the deepest layer and decrease toward the surface.

We will now compare characteristics of the five-layer vortex described above with the corresponding characteristics of a homogeneous vortex of the fourth order \( (N = 4) \), which can be described using the solutions of Rubino et al. (1998). The two vortices are chosen to be “equivalent” in the sense that they possess the same radial distribution of the mean azimuthal velocity. To
this purpose we first determine the radial distribution of the mean azimuthal velocity of the stratified vortex. We then construct a polynomial fit of the seventh degree to such a distribution and finally we use the obtained coefficients to determine the shape of the corresponding (equivalent) homogeneous vortex. A comparison between this shape and the deepest interface of the stratified vortex will be used as a measure of the difference between the two solutions.

According to Rubino et al. (1998), the solutions for homogeneous vortices of the fourth order can be written in the following form:

\[
\begin{align*}
\frac{u}{H_{11005}} &= \frac{\gamma f}{2} \cos(f t + \varphi) r, \\
\frac{v}{H_{9253}} &= -\frac{f}{2} r + \sum_{i=1}^{N} \frac{l_i}{[1 + \gamma \sin(f t + \varphi)]} r^{2i-1}, \\
n &= \sum_{i=0}^{2N-1} \frac{c_i}{[1 + \gamma \sin(f t + \varphi)]^i} r^{2i},
\end{align*}
\]

(19)

and

(20)

where \(N = 4\). The coefficients \(l_i\) and \(c_i\) are related by the relations

\[
\begin{align*}
c_1 &= -\frac{(1 - \gamma^2)f^2 - 4l_1^2}{8ge}, \\
c_2 &= \frac{l_1l_2}{2ge}, \\
c_3 &= \frac{2l_1l_3 + l_2^2}{6ge}, \\
c_4 &= \frac{l_1l_4 + l_2l_3}{4ge}, \\
c_5 &= \frac{2l_2l_4 + l_3^2}{10ge}, \\
c_6 &= \frac{l_1l_5}{6ge}, \quad \text{and} \quad c_7 = \frac{l_2^2}{14ge}.
\end{align*}
\]

(21)

Given the parameters indicated above, we thus obtain

\[
\bar{\rho} = \frac{\sum_{j=1}^{n} \rho_j \int_{0}^{r_h} rh_j \, dr}{\sum_{j=1}^{n} \int_{0}^{r_h} rh_j \, dr}.
\]

(22)

The radial distribution of the mean azimuthal velocity in the stratified vortex is given by

\[
\frac{v}{H_{9253}} = \frac{\sum_{j=1}^{n} V_j}{\sum_{j=1}^{n} \int_{0}^{r_h} h_j \, dr}.
\]

(22)

This distribution, together with its polynomial approximation of the seventh degree, is shown in Fig. 4a. The
coefficients of the fitted polynomial are \( l_1 = 1.362 \times 10^{-5} \) s\(^{-1}\), \( l_2 = 5.071 \times 10^{-17} \) m\(^2\) s\(^{-1}\), \( l_3 = -1.876 \times 10^{-26} \) m\(^4\) s\(^{-1}\), \( l_4 = 2.200 \times 10^{-35} \) m\(^6\) s\(^{-1}\). Inserting these coefficients in (21) we calculate the shape \( h_{\text{hom}} \) of the equivalent homogeneous vortex. The relative difference \( \Delta = (h - h_{\text{hom}})/h \) between the deepest interface \( h \) of the stratified vortex and \( h_{\text{hom}} \) is depicted in Fig. 4b. The noticeable difference between the two shapes, which emerges especially at the vortex periphery, accounts for the role played by the vertical stratification in determining the vortex mean characteristics.

5. Solutions for stratified intermediate lenses

The solutions presented in section 3 can be easily adjusted to describe circular, stratified intermediate (meddy like) frontot vortices (Fig. 5). In order for the reduced-gravity approximation to be valid, the fluid overlying \((j = 0)\) and underlying \((j = n + 1)\) the intermediate vortex has to be considered as motionless. The right-hand sides of (1)–(3) can be written in this case as follows:

\[
F_j = -\gamma \sum_{k=1}^{j-1} (s_{k,j} - s_{k,n+1} - q_{k,j}) \frac{\partial h_k}{\partial r} - \gamma \sum_{k=j}^{n} (1 - s_{k,n+1}) \frac{\partial h_k}{\partial r},
\]

where

\[
2\gamma \sum_{k=1}^{j-1} (s_{k,j} - s_{k,n+1} - q_{k,j})c_k + \gamma \sum_{k=j}^{n} (1 - s_{k,n+1})c_k = -\delta_i \frac{1}{4} (1 - \gamma^2)f^2 + \sum_{m=1}^{i} l_{m,i}(\varphi_{i-m+1}), i = 1, \ldots, 2N - 1,
\]

where, as in the case of surface vortices, \( l_{j,i} = 0 \) if \( i > N \). The circle \( r = r_b \) is the common lateral boundary for all layers (see Fig. 5). Because of this constrain, the coefficients of the solutions have to satisfy \( n \) additional equations from (15), while \( \gamma, \varphi, c_{i,j} \) (or \( l_{j,i} \)), and \( W \) are free parameters of the analytical solution.

To elucidate possible dynamics of a stably stratified, intermediate lens-like vortex, let us consider a five-layer intermediate lens \((n = 5)\) of the fourth order \((N = 4)\) for the following set of parameters: \( \rho_0 = 1031.4 \) kg m\(^{-3}\), \( \rho_1 = 1031.5 \) kg m\(^{-3}\), \( \rho_2 = \rho_1 + \delta(j - 1) \) for \( j = 2, \ldots, n \), \( \rho_{n+1} = 1032.0 \) kg m\(^{-3}\), \( \delta = (\rho_{n+1} - \rho_1)/n \), \( \gamma = 0.2 \), \( W = 5 \times 10^4 \) m, and \( f = 0.86 \times 10^{-4} \) s\(^{-1}\).

In Fig. 6 are depicted, as functions of the radial coordinate, the resulting azimuthal velocity and the corresponding thickness of each layer for a selected time within the inertial period, together with the corresponding quantities of the first-order solution (Rubino and Dotsenko 2006). In each layer, the maximum azimuthal velocity is not found at the eddy periphery (Fig. 6a), as it would be in the first-order stratified solution (Rubino and Dotsenko 2006). Moreover the amplitude of the azimuthal velocity is minimum in the intermediate layer and increases toward the surface and toward the bottom, which corresponds to a well-known characteristic of observed intermediate geophysical vortices.
(see, e.g., Armi and Zenk 1984; Richardson et al. 2000; Budéus et al. 2004). Note also that, as a consequence of the high-order polynomial structure of the azimuthal velocity field, in each layer the distribution of the layer thickness significantly deviates from the paraboloidal one (Fig. 6b).

6. Conclusions

In the present study we discussed a new set of analytical solutions describing the evolution of circular, surface, as well as intermediate lens-like geophysical stratified vortices. The new solutions are no longer restricted to the description of vortices having linear azimuthal velocity and, in addition, they refer to arbitrary, stably stratified features. As a result, a larger degree of realism is achieved than was possible using previous analytical solutions. The obtained solutions comprise, as a subset, the pulson solution (Cushman-Roisin 1987; Rogers 1989), the set of solutions presented by Rubino et al. (1998), as well as those described by Rubino and Dotsenko (2006). They seem to represent the most general analytical features achievable, in the context of the nonlinear reduced-gravity shallow-water equations on an f plane, which refer to vertically stratified circular vortices having linear radial velocity field. To elucidate aspects of the novelty of the proposed solutions, examples were presented referring to a vertically stratified surface vortex and to a vertically stratified intermediate, “meddy-like” vortex, both characterized by azimuthal velocity fields that are nonlinear functions of the radius and by layer shapes that largely deviate from paraboloidal shapes. First, a solution describing a five-layer surface “warm core” eddy was discussed: Within each layer, the obtained azimuthal velocity does not reach its maximum at the eddy periphery, as it would be in the case of the pulson. Moreover, it is largest in the surface layer and decreases monotonically toward the interior vortex layers, which is a characteristic often observed in geophysical surface vortices. Significant differences were also found between the shape of the five-layer vortex and that of its homogenous counterpart having the same mean azimuthal velocity structure. Second, a solution referring to a five-layer intermediate meddy-like vortex was analyzed: In each layer, the obtained azimuthal velocity is maximum in the interior part of the vortex and decreases toward its center and its periphery. Its magnitude is largest in the intermediate layer and decreases toward the surface and toward the bottom, which are characteristics resembling characteristics of observed intermediate geophysical vortices.

Analytical solutions of complex geophysical problems in a rotating, stratified environment like the ones presented here (which, moreover, show fundamentally a “simple” structure), apart from their relevance in explicating fundamental aspects of observed dynamics and their utility in testing the performances of 3D numerical model, possess a special fascination: they add a further step in the attempt to reconcile abstract possibilities with physical reality.

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APPENDIX
Solution of the Second Order for Two-Layer Surface Vortices

In the case of surface vortices having two active layers \((n = 2)\), the second-order solution \((N = 2)\) can be expressed in the following form [see (16)–(18)]:

\[
\begin{align*}
    u_1 &= \frac{\gamma f}{2} \cos(f t + \varphi) r + \frac{l_{1,1} r^3}{1 + \gamma \sin(f t + \varphi)^2}, \\
    u_2 &= \frac{\gamma f}{2} \cos(f t + \varphi) r, \\
    h_1 &= \frac{c_{1,0}}{1 + \gamma \sin(f t + \varphi)^2} + \frac{c_{1,1} r^2}{1 + \gamma \sin(f t + \varphi)^2}, \\
    h_2 &= \frac{c_{2,0}}{1 + \gamma \sin(f t + \varphi)^2} + \frac{c_{2,1} r^2}{1 + \gamma \sin(f t + \varphi)^2}.
\end{align*}
\]

and

\[
\begin{align*}
    c_{1,0} &= -c_{1,1} W^2 - c_{1,2} W^4 - c_{1,3} W^6, \\
    c_{1,1} &= \frac{\ell_{1,1}^2 - \ell_{1,2}^2}{2 g e_1}, \\
    c_{1,2} &= \frac{l_{1,1} l_{1,2} - l_{2,1} l_{2,2}}{2 g e_1}, \\
    c_{1,3} &= \frac{\ell_{1,1} - \ell_{2,1}}{2 g e_1}, \\
    c_{2,0} &= -c_{2,1} W^2 - c_{2,2} W^4 - c_{2,3} W^6, \\
    c_{2,1} &= \frac{(1 - \gamma^2) f^2}{8 g e_2} - \frac{(e_0 - e_1) l_{1,1}^2 - e_2^2}{2 g e_1 e_2}, \\
    c_{2,2} &= \frac{e_0 l_{1,2} l_{2,2} - (e_0 - e_1) l_{1,1} l_{1,2}}{2 g e_1 e_2}, \text{ and} \\
    c_{2,3} &= \frac{e_0 l_{2,2}^2 - (e_0 - e_1) l_{1,2}^2}{6 g e_1 e_2},
\end{align*}
\]

where \(e_0 = 1 - \rho_1/\rho_s, e_1 = 1 - \rho_1/\rho_2, e_2 = 1 - \rho_2/\rho_s\), and \(W\) represents the radius of the outcropping line at the sea surface for \(t = -\epsilon f\). Note that \(\varphi, \gamma,\) and \(l_{i,j}\) are free parameters of the solution.

REFERENCES


