Welfare Implications of Public Debt Denomination in a Small Open Economy* (Preliminary)

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Abstract

The aim of the paper is to discuss the welfare implications of the choice between nominal and indexed debt in a small open economy. Contrary to what happens in a closed economy, indexed debt may be preferable in this context. The reason is that, in an open economy (namely in a small open economy in the context of a monetary union), inflation variability is induced exogenously, and thus may be unfavourable for nominal debt state-contingent properties.

I propose a model for a small open economy with exogenous monetary policy to discuss this question. I conclude that the choice between indexed and nominal debt may have significant implications to the conduct of optimal public debt policy, though the welfare gain/loss of nominal debt is negligible.

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1. INTRODUCTION

This paper is interested in discussing the optimal public debt structure for a small open economy. Specifically, we are concerned with the discussion between nominal and indexed debt, which has regained interest with the recent issuance of inflation-linked bonds (ILB) in several euro area countries (Italy, Greece, and Germany are the most recent examples), following others that have used this type of debt for a longer period (UK, Australia, Canada, Sweden, US, or France). The following table gives an account of the increasing importance of this type of debt in OECD countries.

Table 1 - Share of Inflation-Linked Debt (as % of total debt)

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</thead>
<tbody>
<tr>
<td>Australia</td>
<td>...</td>
<td>...</td>
<td>1.2</td>
<td>6.0</td>
<td>11.8</td>
</tr>
<tr>
<td>Canada</td>
<td>...</td>
<td>...</td>
<td>3.0</td>
<td>1.9</td>
<td>5.4</td>
</tr>
<tr>
<td>France</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>0.8</td>
<td>16.0</td>
</tr>
<tr>
<td>Germany</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>1.0</td>
</tr>
<tr>
<td>Greece</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>7.0</td>
</tr>
<tr>
<td>Italy</td>
<td>...</td>
<td>0.2</td>
<td>...</td>
<td>...</td>
<td>4.9</td>
</tr>
<tr>
<td>New Zealand</td>
<td>...</td>
<td>...</td>
<td>2.4</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>Sweden</td>
<td>...</td>
<td>...</td>
<td>2.5</td>
<td>6.5</td>
<td>17.1</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>0.9</td>
<td>6.6</td>
<td>12.0</td>
<td>16.4</td>
<td>25.4</td>
</tr>
<tr>
<td>United States</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>2.5</td>
<td>8.4</td>
</tr>
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Sources: Missale (1999), Falcetti and Missale (2000), and national debt management agencies.

While factors related with the demand side of the government debt market may be in the root of an explanation for this recent trend,¹ I focus here on the discussion

¹With the advent of the euro in the late 90s, euro area government debt markets became increasingly integrated, with market participants regarding bonds issued by different countries as almost
around the implications of such a choice to the conduct of the optimal fiscal policy. Thus, the paper is of a normative nature, rather than a positive one.

This discussion has mainly focused on two inter-related arguments. First, it is argued that nominal debt is prone to inflation surprises by the monetary authority, which in this way is able to reduce the real value of government’s liabilities. Thus, credibility arguments seem to favour the issuance of price-indexed debt, as argued in Lucas and Stokey (1983), or Díaz-Giménez, Giovannetti, Marimon and Teles (2006).

Second, the fact that the real value of nominal debt is dependent on the realization of inflation may benefit nominal debt, depending on the correlation of inflation with other sources of uncertainty on the economy. The idea is that with a proper choice of the debt structure, the government may be able to improve tax smoothing across states of nature. For instance, if inflation and government expenditures are positively related, then periods of high spending, are accompanied by a low real return of nominal debt, which reduces the need for a tax rate increase. I will call this the state-contingency argument. In a Ramsey taxation framework with flexible prices, such as that developed in Chari, Christiano and Kehoe (1991), it is shown that the optimal fiscal policy performance is indeed enhanced with the introduction of nominal debt.

However, both these arguments have been discussed assuming a closed economy context. Well, in a small open economy, especially in one that is tied to a monetary union or some kind of exchange rate peg, these arguments may not be operating. First, credibility of monetary policy is no longer a relevant issue, since this is defined at a supra-national level. Second, for the same reason, inflation shocks are exogenously determined, hence these may be correlated differently than what is endogenously perfect substitutes. The issuance of inflation-linked bonds, and for that matter of extra-long-term bonds (some countries have now issuances in the 50-years segment), may be seen as an effort for distinguishing among other issuers, in a context of increasing interest of institutional participants (such as pension funds) for these types of assets.

For this reason, I set up a small open economy model, where the real return of nominal debt is stochastic and may be correlated with other sources of uncertainty in the economy. Following the more recent literature, I solve for the optimal Ramsey policy in this economy and discuss its properties under alternative debt structures and different correlations between inflation and other sources of uncertainty. With standard preferences it is shown that the optimal allocations of consumption and labour are extremely smooth in any scenario. However, the optimal fiscal policy is significantly altered. Namely, when productivity and inflation are positively correlated and the government is using only nominal debt, public debt is on average much larger, leading to slightly lower consumption, than in the opposite case of a negative correlation between productivity and inflation.

Nevertheless, the comparison between indexed and nominal debt lead us to conclude that this choice does not have significant welfare implications.

The structure of the paper is as follows. In section 2 the literature on optimal public debt management, and the discussion between nominal and indexed debt in particular, is broadly revised. In section 3 I set up a simple model for a small open economy and solve for the conditions that define a Ramsey equilibrium in this setting. In section 4 I solve the model numerically, which allows us to have a better understanding of the properties of the Ramsey allocation in this context and serves the interest of making welfare comparisons between issuing nominal or indexed debt. Section 5 concludes and sets the path for future research.

\footnote{This feature is well-established in the RBC literature for small open economies. In this context the country uses the ability to trade in external financial markets to smooth consumption over time and across states of nature.}
2. LITERATURE REVIEW

The discussion around the relevance of public debt structure was decisively influenced by the work of Barro in the 1970s: first by the formalization of the Ricardian equivalence hypothesis, and later on by establishing tax smoothing over time as an optimal policy. In fact, as suggested in Barro (1997), one can think of optimal debt management in three stages:

1. Ricardian Equivalence is satisfied (implying, among other things, that lump sum taxes are available)\(^3\) – in this world, the difference between tax-financing and debt-financing is irrelevant, since individuals will save any increase of the latter, as they foresee future tax increases will be borne out to repay the debt.

2. Taxes are distortionary – in this case, the authorities should prefer to smooth taxes over time,\(^4\) hence issuing more debt in higher public spending periods (e.g. wartime periods, recessions) and less during booms; nevertheless the debt structure still does not matter in this case.

3. Uncertainty is introduced – in this case, along with tax smoothing over time, the authorities will be concerned with the relations between public debt returns and shocks in macroeconomic variables, and will choose a debt structure that insulates them from significant changes induced by the business cycle: the authorities will choose a debt structure that smooths tax rates across states of nature, as well as over time.

This last result was formally established in Bohn (1990), where the optimal public debt structure was also explicitly defined. Bohn’s formula depends on the covariance

\(^3\)See Barro (1974), where the author introduces an OLG model to show that, even with finitely lived individuals, it is possible to have neutrality of bond-financed deficits.

\(^4\)In Barro (1979) ad-hoc tax distortions are imposed in an infinitely lived representative agent model, to establish the result of tax smoothing over time.
of debt returns with each source of uncertainty (say output and government spending). According to this prescription, the weight of a certain type of debt should be higher, when its real return has a positive covariance with income and negative with public expenditure.

In terms of the discussion between nominal and indexed debt, this means that nominal debt is preferred if the covariance between output and inflation is negative and the covariance between government expenditure and inflation is positive.

<table>
<thead>
<tr>
<th>Nominal is better if</th>
<th>Indexed is better if</th>
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<tr>
<td>$\text{Cov}(Y_t; \pi_t) &lt; 0$</td>
<td>$\text{Cov}(Y_t; \pi_t) &gt; 0$</td>
</tr>
<tr>
<td>$\text{Cov}(G_t; \pi_t) &gt; 0$</td>
<td>$\text{Cov}(G_t; \pi_t) &lt; 0$</td>
</tr>
</tbody>
</table>

In a related paper, Bohn (1988) argues in favour of nominal debt. He introduces shocks to preferences in an OLG monetary model – with cash-credit goods, as in Lucas and Stokey (1983), and ad-hoc distortionary taxes, as in Barro (1979) – and establishes that some nominal debt is always optimal. This stems from the fact that positive shocks to preferences induce more inflation, which in turn depreciates the value of nominal debt.\footnote{Bohn extends the model to include shocks to government spending, as well as to productivity. He concludes that inflation should be positively related with the former, and negatively related with the latter, thus favouring the issuance of nominal debt (see Table 2).}

The approach taken in this literature has in common the fact that tax distortions are imposed by an ad-hoc convex function.\footnote{For a more thorough review of both theoretical and empirical papers following this strand of the literature I refer to Missale (1999).} In “strictly” general equilibrium models, on the contrary, this is usually introduced by labour income taxation, which distorts labour-leisure choices.
The seminal paper on this strand of the literature is Lucas and Stokey (1983), which applies Ramsey’s (1927) optimal taxation theory to the study of fiscal and monetary policy. The analysis begins with a simple barter economy, without capital, where the solution of the Ramsey problem gives rise to the usual tax smoothing result. The authors also show that the optimal solution can be made time-consistent through the use of the maturity structure.

In a stochastic monetary economy (introduced by the existence of cash goods that are subject to a cash-in-advance constraint), however, it is shown that the optimal policy may not be time-consistent, as the government has an incentive to inflate away nominal debt. For this reason the paper presents a strong case against nominal debt.

More recently, Díaz-Giménez, Giovannetti, Marimon and Teles (2006) have argued that nominal debt is a burden on monetary policy, again because it creates a time inconsistency problem, even when the monetary authority does not inflate away the outstanding of debt in equilibrium.7

However, as I have already argued, these considerations are not crucial in the small open economy case. In our case in particular I will argue that the optimal policy is actually time consistent, so we will only focus on state-contingency arguments.

Chari, Christiano and Kehoe (1991) have analyzed the quantitative features of Lucas and Stokey’s (1983) model. Four main findings are summarized, among which is the fact that, in the Lucas and Stokey’s monetary economy with nominal debt only,

7The credibility issue is probably the most cited argument in favour of ILB. In Favero, Missale and Piga (2000), three advantages of indexed debt are briefly discussed: (i) it provides insurance against inflation risk; (ii) it provides a good measure of inflation expectations, thus benefiting the conduction of monetary policy; and (iii) it gives an incentive for low inflation. In the general equilibrium framework, (i) and (iii) are roughly condensed in Lucas and Stokey’s credibility argument, as agents need inflation insurance, only to the extent that monetary authorities have incentives to give rise to surprise inflation. The other seems of more practical relevance.
monetary policy is countercyclical with respect to technology shocks and procyclical with respect to government consumption. Again this raises the same argument as Bohn (1988) in favour of nominal debt (see Table 2).

More recently, Cosimano and Gapen (2003) made a quantitative assessment of these results, by calibrating a model similar to Chari, Christiano and Kehoe (1991)’s to the US economy, and running simulations under various debt-to-income ratios and differing compositions of nominal and indexed debt. They conclude that the welfare gain from using nominal debt to hedge against shocks is large, which is explained by the positive response of inflation, to negative productivity shocks, and to positive government spending innovations (again see Table 2).

A striking feature of Chari, Christiano and Kehoe (1991) results is that the variance of inflation is extremely high. This may imply important distortions in the economy, as suggested by Schmitt-Grohé and Uribe (2004) and Siu (2004). In these two independent papers, the optimal fiscal and monetary policy is conducted in a setting where prices are sticky. It is then shown that the optimal variance of inflation is zero, which eliminates the benefits from the use of nominal debt.

Nevertheless, Correia, Nicolini and Teles (2002) allow for the presence of other policy instruments and argue that the state-contingency properties of nominal debt may be obtained with consumption taxes, and thus without the need of high variability of inflation. Then, with this argument, nominal debt regains relevance as a possible shock-absorber.

Another difficulty with Chari, Christiano and Kehoe’s (1991) line of argument is offered by Aiyagari, Marcet, Sargent and Seppälä’s (2002) analysis of optimal fiscal policy without state-contingent debt. In this paper, the government is restricted to the use of one-period real non-contingent debt. They show that the optimal allocation of tax rates and debt in this incomplete markets setting resembles closely Barro’s (1979) unit root property, which is in contrast with Lucas and Stokey’s (1983) complete
markets story. More interestingly, however, it is shown that in some important cases the welfare loss of losing the state-contingency of government debt is negligible.

This is challenging for the general approach taken here. Nevertheless I believe that it is still relevant to discuss the state-contingency properties of nominal vs. indexed debt, particularly in the case of a small open economy. In fact, while the analysis of a closed economy setting suggests that, based on state-contingency arguments, if something is better it must be nominal debt,8 in a small open economy nominal debt may actually be welfare-reducing, depending on the stochastic properties of inflation.

3. THE MODEL

The approach adopted here is closely related to that taken in Lucas and Stokey (1983) or Chari, Christiano and Kehoe (1991), only we apply it to the small open economy case, where the real return on nominal debt is exogenous and stochastic.

The aim is to close the gap between this micro-founded line of research and the optimal debt management analysis based on ad-hoc distortionary taxes taken in Bohn (1990) and elsewhere.

The economy is populated by an infinitely-lived representative agent, who seeks to maximize her utility. The agent must satisfy a very simple budget balance constraint, where he consumes all his current after-tax income. The absence of financial markets for the private sector is a simplifying assumption that allows us to solve the model numerically using standard dynamic programming methods, while not affecting the relevance of the discussion between government debt instruments. Furthermore, as will be clear from the numerical simulations, this does not preclude the possibility of smoothing consumption over time and across states of nature, which here will be

8 Though the welfare gain may be small, as suggested by Aiyagari, Marcet, Sargent and Seppälä (2002).
offered by the government’s active management of public debt.

The government is required to satisfy a period by period budget constraint, where it must finance an exogenous stream of irrelevant government expenditures using labour income taxes, one-period real (indexed) government debt, or one-period nominal government debt. The government is benevolent and thus maximizes households’ utility.

Households

The preferences of households are summarized by:

$$E_0 \left[ \sum_{t=0}^{\infty} \beta^t U(C_t, 1 - L_t) \right],$$

where $U(c, x)$ is the period utility function satisfying the usual regularity conditions (continuity, twice-differentiability, concavity), $C_t$ is consumption, $X_t = 1 - L_t$ is leisure time, and $L_t$ is the number of hours worked.

The budget constraint (as explained assuming "hand-to-mouth consumers") is as follows:

$$C_t = Z_t(1 - \tau_t) L_t,$$

where $Z_t$ is the exogenous and stochastic productivity level, and $\tau_t$ is the labour income tax rate. In this constraint it is implicit the assumption of a linear production function (i.e. $Y_t = Z_t L_t$), which justifies the fact that the real wage equals the productivity level.

Then, the FOC of the household’s problem is simply:

$$U_X(C_t, 1 - L_t) = Z_t(1 - \tau_t) U_C(C_t, 1 - L_t),$$

where as usual $U_i$ denotes the first derivative of $U$ with respect to variable $i$. 
Government

The fiscal authority has to finance an exogenous stochastic stream of public expenditure, using proportional labour income taxes and two alternative types of debt: real non-contingent \((I_t)\), which pays a constant real return \(R^*\); or nominal non-contingent debt \((N_t)\), which pays an exogenous and stochastic real return \(R_t\):

\[
G_t + R^* I_{t-1} + R_t N_{t-1} = \tau_t Z_t L_t + I_t + N_t. \tag{4}
\]

The real return of each type of debt has the following interpretation (where \(r^N\) is the constant nominal interest rate):

\[
R^* = 1 + r^N - E_t[\pi_{t+1}], \tag{5}
\]

\[
R_{t+1} = 1 + r^N - \pi_{t+1}. \tag{6}
\]

Hence, the difference between the two is simply the unanticipated inflation surprise \(R_{t+1} - R^* = -\pi_{t+1} = - (\pi_{t+1} - E_t[\pi_{t+1}])\).

By concentrating on \(R^*\) and \(R_{t+1}\) we abstract from modelling inflation explicitly. Apart from the gain in model tractability, this has the advantage of leaving open the possibility to reinterpret \((I_t, N_t)\) as other debt instruments, such as domestic- vs. foreign-currency debt, or short- vs. long-term debt. Naturally, in that case one would also need to reinterpret \(R_{t+1}\) and redefine equation (6).

From the two equations above one immediately sees that a no-arbitrage condition arises: \(R^* = E_t[R_{t+1}]\). Moreover, in order to have a steady state we will also assume that \(R^* = 1/\beta\).

Ramsey equilibrium

A competitive equilibrium is completely described by equations (2), (3), and (4).
The *Ramsey equilibrium* is defined as the competitive equilibrium that maximizes households’ utility:

\[
\max_{\{C,L,I,N\}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t U(C_t, 1 - L_t) \right] \tag{7}
\]

subject to (2), (3), and (4)

\[I_{t-1}, N_{t-1} \text{ given}; I_t, N_t \in \Delta \subset \mathbb{R}^2,\]

where \(\Delta\) is a limited set defined to guarantee that the government will always repay its debts and will never enter in a Ponzi scheme (i.e. debt is risk free).\(^9\)

**Primal approach**

As is usual in this type of settings, it will be useful to recast the Ramsey problem using the primal approach as in Lucas and Stokey (1983).

To do this, one simply uses the household’s FOC (3) to substitute for the tax rate in the household’s and government’s budget constraints, equations (2) and (4) respectively. Then we have a simpler problem, which is characterized only on the allocations of consumption, labour and public debt:

\[
\max_{\{C,L,I,N\}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t U(C_t, 1 - L_t) \right] \tag{8}
\]

subject to

\[U_C(C_t, 1 - L_t) C_t = U_X(C_t, 1 - L_t) L_t \tag{9}\]

\[C_t + G_t + R^* I_{t-1} + R_t N_{t-1} = Z_t L_t + I_t + N_t \tag{10}\]

\[I_{t-1}, N_{t-1} \text{ given}; I_t, N_t \in \Delta\]

\(^9\)An assumption also followed in Aiyagari, Marcet, Sargent and Seppälä (2002), for instance. Moreover, it will be useful for computational purposes to limit the possible values of the state variable (public debt) to a finite grid of points.
Equation (9) is analogous to Lucas and Stokey’s (1983) implementability constraint, only now it must be satisfied in every period (and not only in present value terms), since we have "hand-to-mouth" consumers. Equation (10), on the other hand, is the economy’s resource constraint in an open economy case, where the possibility to trade with the rest of the world is captured by the evolution of the external debt (which in our simple case equals the public debt). Also notice that the optimal path of the tax rate can then be easily recovered using again equation (3).

Denoting the Lagrange multipliers on the above restrictions by $\psi_t$ and $\lambda_t$, respectively, we have the following FOC for this problem:

\begin{align}
U_{C,t} + \psi_t[U_{C,t} + U_{CC,t}c_t - U_{XC,t}L_t] + \lambda_t &= 0, \\
U_{X,t} + \psi_t[U_{X,t} - U_{XX,t}L_t + U_{CX,t}c_t] + \lambda_tZ_t &= 0,
\end{align}

\begin{align}
\lambda_t &= E_t[\lambda_{t+1}], \\
\lambda_t &= \beta E_t[\lambda_{t+1}R_{t+1}].
\end{align}

An interesting feature of our set up is that, although markets are not complete, the solution is time consistent. To see this one just needs to observe that the FOC of the problem do not depend on the initial state of the economy $I_{-1}, N_{-1}$.

The intuition for this is quite simple: since interest rates are exogenous, the government is not able to manage this in its interest by changing between tax- and debt-financing (contrary to the closed economy case, when interest rates are endogenous and may then be manipulated by the fiscal authority, as shown by Lucas and Stokey (1983)).
Recursive formulation

The problem can also be written in a recursive manner, which will allow us to solve numerically for the optimal allocations. However, we will not be able to pin down the optimal path of both indexed and nominal debt, since we have only one condition (the resource constraint) governing the law of motion of public debt. Hence, the approach taken here (following for instance Missale (1999) or Cosimano and Gapen (2003)) is to fix in advance the composition of public debt between the two instruments and compare the optimal allocations and the respective welfare under different debt structures (namely under the two extreme cases: 100% indexed debt vs. 100% nominal debt).

Denoting the share of indexed debt by $h$, and the total level of public debt at time $t$ by $D_t$, we have $I_t = hD_t$ and $N_t = (1-h)D_t$. The resource constraint can then be re-written as

$$C_t + G_t + R^*hD_{t-1} + R_t(1-h)D_{t-1} = Z_tL_t + D_t.$$  \hspace{1cm} (15)

Then, we are able to write the Bellman equation of our recursive problem:

$$V(D; S) = \max \{ U(C, 1 - L) + \beta E[V(D'; S')] \} \hspace{1cm} (16)$$

s. to $U_C(C, 1 - L)C = U_X(C, 1 - L)L$

$C + G + R^*hD + R(1-h)D = ZL + D'$

$D' \in \Delta = [D_{\min}, D_{\max}]$

where $S$ denotes the vector of exogenous and stochastic state variables, which in our case is $S = (Z, G, R)$. As usual, primes ($x'$) denote variables one-period ahead.

In section 4 we will proceed to calibrate this problem and solve it numerically. However, in the following sub-section, I will consider an alternative approach to solve
the Ramsey problem, which will be useful to establish a link between this kind of approach on one side, and the approach relying on ad-hoc distortionary taxation à la Barro, such as Bohn (1990), on the other. As will be shown, in a simple example we can replicate Bohn’s propositions. The reader more interested in the numerical results obtained from the previous approach, however, may wish to skip this and go directly to section 4.

**Dual approach - an example re-affirming Bohn (1990)**

In our assumedly simple framework, the households’ preferred allocation of consumption and labour at each period and state, given the chosen tax rate, can be obtained by solving equations (2) and (3). Denoting this solution by \( C(\tau_t; Z_t) \) and \( L(\tau_t; Z_t) \) and defining the corresponding indirect utility function as \( W(\tau_t; Z_t) = U(C(\tau_t; Z_t); 1 - L(\tau_t; Z_t)) \), one can rewrite the Ramsey problem synthetically as

\[
\max_{\{\tau, I, N\}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t W(\tau_t; Z_t) \right] \tag{17}
\]

s. t. \( G_t + R^t I_{t-1} + R_t N_{t-1} = Z_t \tau_t L_t(\tau_t; Z_t) + I_t + N_t \) \( \tag{18} \)

\[ I_{-1}, N_{-1} \text{ given; } I_t, N_t \in \Delta \]

The FOC of this problem are the following:

\[
W_{\tau,t} = [Z_tL_t(\tau_t; Z_t) + Z_t\tau_tL_{\tau,t}]\theta_t
\]

\[
\theta_t = E_t[\theta_{t+1} + 1]
\]

\[
\theta_t = \beta E_t[\theta_{t+1}R_{t+1}]
\]

15
Or simply,

\[
\frac{W_{\tau,t}}{Z_t L_t(\tau_t; s_t) + Z_t \tau_t L_{\tau,t}} = E_t \left[ \frac{W_{\tau,t+1}}{Z_{t+1} L_{t+1}(\tau_{t+1}) + Z_{t+1} \tau_{t+1} L_{\tau,t+1}} \right],
\]

(19)

\[
\frac{W_{\tau,t}}{Z_t L_t(\tau_t; s_t) + Z_t \tau_t L_{\tau,t}} = \beta E_t \left[ \frac{W_{\tau,t+1}}{Z_{t+1} L_{t+1}(\tau_{t+1}) + Z_{t+1} \tau_{t+1} L_{\tau,t+1}} R_{t+1} \right],
\]

(20)

where \( \theta_t = W_{\tau,t}/(Z_t L_t + Z_t \tau_t L_{\tau,t}) \), the Lagrange multiplier for the government budget constraint, can be interpreted as the net marginal cost of taxes.

These correspond to equations (6a) and (6b) in Bohn (1990):

\[
h'(\tau_t) = E_t [h'(\tau_{t+1})],
\]

\[
h'(\tau_t) = \beta E_t [h'(\tau_{t+1}) R_{t+1}],
\]

where \( h'(\tau_t) \) is the marginal deadweight loss of taxes. In his case \( h(\tau_t) \) is an ad-hoc function assumed to be quadratic. Hence, these two equations result in two conditions for tax smoothing over time and across states of nature (equations (7a) and (7b)):

\[
E_t [\Delta \tau_{t+1}] = 0,
\]

\[
E_t [\Delta \tau_{t+1} R_{t+1}] = 0.
\]

In contrast, in our case this is endogenously determined. A simple example, however, illustrates how these two equations may be recovered in our setting. Assuming a quasi-linear utility function such as \( U(c, x) = c - \frac{1}{2}(1 - x)^2 \), we have \( L(\tau_t; Z_t) = Z_t(1 - \tau_t) \), \( C(\tau_t; Z_t) = (Z_t(1 - \tau_t))^2 \), and \( W(\tau_t; Z_t) = \frac{1}{2}[Z_t(1 - \tau_t)]^2 \). Hence, in this case, the marginal cost of taxation is
\[ \theta_t = \frac{1 - \tau_t}{2\tau_t - 1}, \]  

which is independent on the productivity shock and can be shown to be a strictly decreasing function of the tax rate.\(^{10}\) Hence, equations (19) and (20) reduce to \( E_t [\Delta \tau_{t+1}] = 0 \) and \( E_t [\Delta \tau_{t+1} R_{t+1}] = 0 \) as in Bohn (1990).

## 4. NUMERICAL ANALYSIS

In this section I report some numerical results obtained for our economy under standard CRRA preferences:

\[ U(C, X) = \frac{C^{1-\sigma_1} - 1}{1 - \sigma_1} + \eta \frac{X^{1-\sigma_2} - 1}{1 - \sigma_2}. \]  

(22)

Our baseline parametrization of the model is the following:

<table>
<thead>
<tr>
<th>( \sigma_1 )</th>
<th>( \sigma_1 )</th>
<th>( \eta )</th>
<th>( \bar{G} )</th>
<th>( \bar{D} )</th>
<th>( D_{\text{min}} )</th>
<th>( D_{\text{max}} )</th>
<th>( \bar{Z} )</th>
<th>( \bar{R} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0.2( \bar{Y} )</td>
<td>0.6( \bar{Y} )</td>
<td>0</td>
<td>2( \bar{D} )</td>
<td>1</td>
<td>1.01</td>
</tr>
</tbody>
</table>

Here \( \bar{G}, \bar{D}, \bar{Z}, \bar{R} \), and \( \bar{Y} \) are the deterministic steady states of government expenditures, public debt, productivity, interest rate, and output, respectively.

As discussed, the recursive formulation allows us to use an iterative algorithm on the value function defined by the Bellman equation, which enables us to solve for the policy function of public debt.\(^{11}\)

\(^{10}\)Provided we calibrate the model in such a way that we always have \( \tau_t > \frac{1}{2} \) or \( \tau_t < \frac{1}{2} \).

\(^{11}\)The algorithm follows standard stochastic dynamic programming methods. For a reference, see Ljungqvist and Sargent, chapter 3. A finite grid of 101 points is defined for the endogenous state variable, which in our case is limited between 0 and 2 times the deterministic steady state value (i.e. public debt is limited between 0 and 120% of average GDP).
I conduct two different experiments, both borrowed from Buera and Nicolini (2004). First I concentrate on productivity shocks, leaving government expenditures fixed at 20% of the deterministic steady state level of output. Second, I do the opposite by fixing the productivity level and letting government spending follow a stochastic process. In any of the two cases and when the government is using nominal debt, the interest rate follows a random path, which is allowed to have different correlations with the other source of uncertainty of the economy.

Each stochastic variable has two possible states of nature (High, Low). Thus we have 4 possible realizations of state $S$ at each date. This is represented as a $4 \times 1$ vector $S = (S^{LL}, S^{LH}, S^{HL}, S^{HH})'$, where the first superscript represents the state of the productivity (or government expenditure) shock and the second the state of the interest rate shock.

The state $S$ follows a Markov process, which is governed by a $4 \times 4$ transition probability matrix. This will be different depending on the correlation between the two shocks. One way to introduce this correlation is to create a parameter $p$ that defines the probability of the two shocks having the same sign. See Appendix A for a detailed description of the construction of these matrices.

I simulate the economy under the benchmark case of constant interest rate (i.e. the indexed debt case), plus several alternative situations (for different correlations between the shocks) with the government using only nominal debt.

4.1. Business cycle example

In this example the productivity shock follows the calibration used in Buera and Nicolini (2004). The unconditional probability of being in a high productivity state is around 0.7 and the expected number of consecutive periods with high productivity is 12 quarters (3 years). For this case, $\pi_{LL} = 0.812$ and $\pi_{HH} = 0.919$. Given this distribution, the possible values of productivity are chosen to target the two
first moments (average equal to 1, and standard deviation around 0.07%). Then, $Z_L = 0.9895$ and $Z_H = 1.00452$.

I start the exposition of the results by analyzing the optimal policy function for the public debt.\footnote{All figures and tables are in Appendix B.1.} Figure 1 gives us the level of public debt chosen for the next period (on the y-axis), given the current level of debt (on the x-axis) and the current state of the economy. This is compared with the 45° line. As one may observe, when the productivity level is low (states $LL$ and $LH$) the public debt increases slightly within one period, and if we stay in the same state it keeps increasing up to the upper limit $D_{\text{max}}$. In contrast, when we are in a high state for productivity, the public debt is always reduced down to the lower limit. This behavior of the public debt resembles a unit root process, a result that is close to Barro’s prediction, and is expected in our incomplete markets setting (as discussed in Aiyagari, Marcet, Sargent and Seppälä (2002)).

When nominal debt is issued, the policy function of public debt crucially depends on the correlation between the two shocks. If this correlation is negative (i.e. the correlation between productivity and inflation is positive), the public debt will almost always tend to increase, even in the high productivity states (see figure 2 for an example). The explanation is that in this case the interest rate pattern is destabilizing, which needs to be financed with higher debt.

When the correlation is positive, on the contrary, the policy function shows a slightly greater tendency to reduce the debt level, when compared to the indexed debt economy, as the interest rate pattern is now stabilizing.

I then simulate the model for 200 periods (50 years), to have a better understanding of how the optimal policy is chosen and how this is affected by the choice of nominal and indexed debt. In figures 4 to 6 we have the simulated paths of productivity, interest rate, public debt, tax rate, consumption, and labour supply and output under
the indexed debt case and two scenarios using nominal debt.

The most striking feature of the model is the smoothness of consumption and labour paths in all scenarios, which at first may seem surprising given our assumption of "hand-to-mouth consumers". This shows that the government is able to insure consumers from exogenous uncertainty, through the use of fiscal policy. In fact, in a small open economy, and given the type of preferences used, this excess smoothness of consumption is not a surprising result.\footnote{See Correia, Neves and Rebelo (1995).}

As one may observe, however, in order to achieve this the government follows radically different public debt policies for different levels of the correlation between productivity and interest rates. When the correlation is negative, the public debt level increases up to the upper bound of public debt and stays relatively stable at that level. On the contrary, when the correlation is positive the public debt level is more prone to be stuck at low levels. This in turn implies a slightly higher average level of consumption in the former case, and a slightly lower one in the latter.

In order to make welfare comparisons properly, I have simulated the economy for a much longer period (1000000 periods) and was then able to closely approximate the value of welfare under different situations. Following Lucas (1987), I estimate the welfare gain, relative to the indexed debt case, measured as the percentage of consumption decrease in every period that would leave consumers indifferent between the case at hand and the benchmark indexed debt case.

The results in Table 4 suggest that the welfare impact of issuing nominal debt is negligible. In fact, in absolute terms the highest difference is around 0.01\% of consumption.\footnote{As an example for comparison, Lucas (1987) estimates the welfare improvement of eliminating the consumption variability to be between 0.008\% and 0.042\% (for $\sigma_1$ equal to 1 and 5, respectively).}

One important feature is that there appears to exist a cut-off level for the correla-
tion between productivity and inflation, above which there is a welfare loss of some 0.01% of consumption, and below which there is a welfare gain of some 0.001% of consumption. While the sign of the welfare effect is consistent with Bohn’s (1990) story (see again Table 2), the fact that this effect is basically constant, given that the correlation is above or below a certain cut-off level hides a crucial difference. In our setting, the result is driven by a difference on the average level of consumption and labour (which are respectively lower and higher, when the correlation is positive), and not on the standard deviation.

This critically depends on the debt limits, so we worked some examples with different debt limits, to understand how this affected the results.

Somewhat surprisingly, with larger debt limits (see Table 5) the difference of public debt policy across the alternative scenarios is even more striking. This is possibly explained by the fact that these limits are now closer to the natural debt limits, as defined by Aiyagari (1994).\textsuperscript{15} Hence, when we reach the limits, it is more probable that we get "stuck" at or close to that level, which leads to a slightly larger impact on average consumption and labour, and thus on welfare.\textsuperscript{16} With narrower debt limits, on the contrary, the welfare effects are smaller (see Table 6).

Another noteworthy feature of the results is that the welfare effect is reversed for extremely low levels (close to \(-1\)) of the correlation.

The model was also solved allowing for significantly different parametrizations (namely with higher risk aversion coefficients, or significantly higher standard devia-

\textsuperscript{15}In a loose sense the natural debt limit would be the highest possible value of public debt, such that the government would always be able to repay it almost surely (i.e. even in the worst outcome). Unfortunately, this is difficult to compute in general.

\textsuperscript{16}The fact that the welfare effect is now slightly positive, when the correlation between productivity and inflation is positive, merely reflects the fact that now, when debt is indexed it increases up to the upper limit, as in the scenarios where the correlation between productivity and inflation are positive.
tion of the shocks). The results are broadly unchanged, so for expositional purposes I opt to omit them here.

4.2. War peace example

This example also follows Buera and Nicolini (2004). A state of war is described by a huge increase in government expenditure (to around 2.5 times the level observed in a state of peace). The transition matrix is defined in such a way that a war probably occurs twice in every century and lasts some 3 years. So we set $\pi_{LL} = 0.995$ and $\pi_{HH} = 0.917$. The level of government expenditure in each state is chosen to target an average value equal to $G$, so we have $G_L = 0.183Y$ and $G_H = 0.483Y$.

In this case, the simulations show a very similar picture in all scenarios: when a war occurs, both public debt and taxes are raised to pay for the higher government expenditure; as soon as the war is over, the tax rate is significantly reduced, while the public debt starts declining only gradually. When the correlation between government expenditures and inflation is negative (a destabilizing situation according to Bohn’s prescription) a war has a more prolonged effect on the economy, but this is hardly noticed in the simulations (see figures 7 to 9).

The results in Table 7 also show that the welfare impact of issuing debt in nominal or real terms are very small (even less significant than in the business cycle example), though again the signs conform to Bohn’s (1990) conclusions.

5. CONCLUSIONS

The analysis herein provides a simple framework to quantify the possible welfare gains of issuing nominal debt in a small open economy. The calibrated examples show that while the optimal policy may be drastically different, the welfare differences are negligible. This is explained by the smoothness of consumption and labour in all
scenarios, which is a common outcome in small open economy models, when one uses standard preferences.

However, this feature is at odds with empirical evidence. This problem has been addressed by Mendoza (1991), or Correia, Neves and Rebelo (1995), who propose different preferences to replicate the stylized facts of a small open economy. Then, a natural robustness test to the results presented in this paper is to solve the model using this type of preferences.

Nevertheless, we are able to show that, depending on the debt limits imposed, the effects of the choice between nominal and indexed debt affect not only the second moments of consumption and labour, but more importantly its average values, which as far as I am aware is a novel result in the literature, and deserves a deeper understanding of this issue.

These results ought to be extended to a richer economy, namely one in which the government has more instruments available (consumption taxes, endogenous government expenditure, transfers), as it does practice.

Furthermore, it would naturally be interesting to have a view on the probable value of these correlations in a "strictly speaking" general equilibrium setting. The idea would be to set up a 2-country currency union model, and discuss the properties of these economies under an optimal behavior of both fiscal authorities. This discussion will be left for future research.

Appendix A. Computing the transition probability matrices

In this Appendix I show how the transition probability matrices were constructed to allow for different correlations between the interest rate and the other shock (technology or government expenditure).

I start by defining the stochastic process for the latter. This is given by a certain $2 \times 2$ transition probability matrix, say
\[ \Pi = \begin{bmatrix} \pi_{LL} & 1 - \pi_{LL} \\ 1 - \pi_{HH} & \pi_{HH} \end{bmatrix}, \] 

(23)

where \( \pi_{jj} \) is the probability of staying in state \( j \), given that we are in state \( j \). This is the matrix governing the stochastic process of productivity (or government expenditures), when this is the only source of uncertainty (as in the indexed debt case).

When we consider a stochastic interest rate, however, we have a Markov process with 4 states. As discussed, it will be relevant to consider different cases for the correlation between the two shocks. I define \( p \) as the probability of the two shocks having the same sign (i.e. \((Z_H, R_H)\) or \((Z_L, R_L)\) in the case of the productivity shock). Then, if \( p \) is 0.5, the correlation between the two shocks is 0. If \( p < 0.5 \) the correlation is negative, and converges to \(-1\) as \( p \) approaches 0; while the opposite happens as \( p \) increases to a value close to 1.

Then, for a chosen value of \( p \) the transition probability matrix of our 4-state Markov process is as follows:

\[ P = \begin{bmatrix} \pi_{LL}p & \pi_{LL}(1 - p) & (1 - \pi_{LL})(1 - p) & (1 - \pi_{LL})p \\ \pi_{LL}p & \pi_{LL}(1 - p) & (1 - \pi_{LL})(1 - p) & (1 - \pi_{LL})p \\ (1 - \pi_{HH})p & (1 - \pi_{HH})(1 - p) & \pi_{HH}(1 - p) & \pi_{HH}p \\ (1 - \pi_{HH})p & (1 - \pi_{HH})(1 - p) & \pi_{HH}(1 - p) & \pi_{HH}p \end{bmatrix}. \] 

(24)

I will consider different values for \( p \), between 0 (i.e. correlation equal to \(-1\)) and 1 (correlation equal to 1).
Appendix B.1. Business cycle example

Fig 1- Policy function of indexed public debt

Fig 2 - Policy function of nominal public debt ($\text{correl}(Z, \pi) = 1$)
Fig 3 - Policy function of nominal public debt $(\text{correl}(Z, \pi) = -0.52)$

Fig 4 - Simulated paths with indexed debt
Fig 5 - Simulated paths with nominal debt \((corr(Z, \pi) = 1)\)

Fig 6 - Simulated paths with nominal debt \((corr(Z, \pi) = -0.52)\)
Table 4 - Welfare comparisons (prod. shock)

<table>
<thead>
<tr>
<th>Type of debt</th>
<th>$p$</th>
<th>$correl(Z, R)$</th>
<th>$correl(Z, \pi)$</th>
<th>$E[W]$</th>
<th>Welfare gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Indexed</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>−251.74</td>
<td>−</td>
</tr>
<tr>
<td>Nominal</td>
<td>0</td>
<td>−1</td>
<td>1</td>
<td>−254.31</td>
<td>−0.0107%</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.1</td>
<td>−0.77</td>
<td>0.77</td>
<td>−254.32</td>
<td>−0.0107%</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.2</td>
<td>−0.57</td>
<td>0.57</td>
<td>−254.35</td>
<td>−0.0109%</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.3</td>
<td>−0.37</td>
<td>0.37</td>
<td>−254.33</td>
<td>−0.0108%</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.4</td>
<td>−0.18</td>
<td>0.18</td>
<td>−254.20</td>
<td>−0.0102%</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>−253.49</td>
<td>−0.0073%</td>
</tr>
<tr>
<td>Nominal</td>
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<td>0.18</td>
<td>−0.18</td>
<td>−251.43</td>
<td>0.0013%</td>
</tr>
<tr>
<td>Nominal</td>
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<td>0.37</td>
<td>−0.37</td>
<td>−251.42</td>
<td>0.0013%</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.8</td>
<td>0.57</td>
<td>−0.57</td>
<td>−251.42</td>
<td>0.0013%</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.9</td>
<td>0.78</td>
<td>−0.78</td>
<td>−251.48</td>
<td>0.0011%</td>
</tr>
<tr>
<td>Nominal</td>
<td>1</td>
<td>1</td>
<td>−1</td>
<td>−252.32</td>
<td>−0.0024%</td>
</tr>
</tbody>
</table>
Table 5 - Welfare comparisons (prod. shock; $D_{\text{min}} = -\mathcal{D}$; $D_{\text{max}} = 3\mathcal{D}$)

<table>
<thead>
<tr>
<th>Type of debt</th>
<th>$p$</th>
<th>$\text{correl}(Z, R)$</th>
<th>$\text{correl}(Z, \pi)$</th>
<th>$E[W]$</th>
<th>Welfare gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Indexed</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>−256.23</td>
<td>−</td>
</tr>
<tr>
<td>Nominal</td>
<td>0</td>
<td>−1</td>
<td>1</td>
<td>−255.95</td>
<td>0.0011%</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.1</td>
<td>−0.77</td>
<td>0.77</td>
<td>−256.05</td>
<td>0.0007%</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.2</td>
<td>−0.57</td>
<td>0.57</td>
<td>−256.09</td>
<td>0.0006%</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.3</td>
<td>−0.37</td>
<td>0.37</td>
<td>−256.07</td>
<td>0.0006%</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.4</td>
<td>−0.18</td>
<td>0.18</td>
<td>−256.04</td>
<td>0.0008%</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>−255.59</td>
<td>0.0026%</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.6</td>
<td>0.18</td>
<td>−0.18</td>
<td>−249.92</td>
<td>0.0261%</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.7</td>
<td>0.37</td>
<td>−0.37</td>
<td>−249.76</td>
<td>0.0268%</td>
</tr>
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<td>Nominal</td>
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<td>0.57</td>
<td>−0.57</td>
<td>−249.73</td>
<td>0.0269%</td>
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<td>Nominal</td>
<td>0.9</td>
<td>0.78</td>
<td>−0.78</td>
<td>−249.78</td>
<td>0.0267%</td>
</tr>
<tr>
<td>Nominal</td>
<td>1</td>
<td>1</td>
<td>−1</td>
<td>−256.22</td>
<td>0.0000%</td>
</tr>
</tbody>
</table>
Table 6 - Welfare comparisons (prod. shock; $D_{\text{min}} = 0.5D$; $D_{\text{max}} = 1.5D$)

<table>
<thead>
<tr>
<th>Type of debt</th>
<th>$p$</th>
<th>$\text{correl}(Z, R)$</th>
<th>$\text{correl}(Z, \pi)$</th>
<th>$E[W]$</th>
<th>Welfare gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Indexed</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–252.44</td>
<td>–</td>
</tr>
<tr>
<td>Nominal</td>
<td>0</td>
<td>–1</td>
<td>1</td>
<td>–253.56</td>
<td>–0.0046%</td>
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<tr>
<td>Nominal</td>
<td>0.1</td>
<td>–0.77</td>
<td>0.77</td>
<td>–253.61</td>
<td>–0.0049%</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.2</td>
<td>–0.57</td>
<td>0.57</td>
<td>–253.62</td>
<td>–0.0049%</td>
</tr>
<tr>
<td>Nominal</td>
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<td>–0.37</td>
<td>0.37</td>
<td>–253.63</td>
<td>–0.0049%</td>
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<tr>
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<td>0.18</td>
<td>–253.58</td>
<td>–0.0047%</td>
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<tr>
<td>Nominal</td>
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<td>0</td>
<td>0</td>
<td>–252.84</td>
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<td>0.18</td>
<td>–0.18</td>
<td>–252.24</td>
<td>0.0008%</td>
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<td>Nominal</td>
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<td>0.37</td>
<td>–0.37</td>
<td>–252.22</td>
<td>0.0009%</td>
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<td>–252.21</td>
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<td>0.0009%</td>
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<tr>
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<td>1</td>
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<td>–252.30</td>
<td>0.0006%</td>
</tr>
</tbody>
</table>
Appendix B.2. War/peace example

Fig 7 - Simulated paths with indexed debt

Fig 8 - Simulated paths with nominal debt ($\text{corr}(G, \pi) = 1$)
Fig 9 - Simulated paths with nominal debt \((\text{correl}(G, \pi) = -0.4)\)

<table>
<thead>
<tr>
<th>Type of debt</th>
<th>(p)</th>
<th>(\text{correl}(G, R))</th>
<th>(\text{correl}(G, \pi))</th>
<th>(E[W])</th>
<th>Welfare gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Indexed</td>
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<td>–</td>
<td>–</td>
<td>–345.97</td>
<td>–</td>
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<td>0.4</td>
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<td>1</td>
<td>–1</td>
<td>–346.08</td>
<td>–0.0004%</td>
</tr>
</tbody>
</table>

**REFERENCES**


